On the global convergence of a class of distributed algorithms for maximizing the coverage of a WSN

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Abstract— **We consider the problem of finding a periodic schedule for the wake-up times of a set of nodes in a Wireless Sensor Network (WSN) that optimizes the coverage of the the nodes are deployed on. An exact solution of the problem entails the solution of an Integer Linear Program and is hardly viable on low power nodes. In this paper, we study the convergence of an efficient decentralized algorithm for node scattering by casting the problem into one of asymptotic stability for a particular class of linear switching systems. We present asymptotic stability results for generic WSN topologies and an application of the algorithm to the coverage problem to show the effectiveness of the proposed solution.**

I. INTRODUCTION

Wireless Sensor Networks (WSN) are today increasingly employed as monitoring and active devices in a wide spectrum of different applications that require processing capabilities to recognize, classify and react to external events. In particular, *safety-critical* applications, such as security, structural monitoring and critical process control, derive many benefits from WSNs. One important issue to be addressed in these contexts is the ability to preserve the functionality and integrity of the network countering device failures, e.g, due to power supply exhaustion, or by structural changes. These effects can be mitigated by adjusting the network parameters for maximum operational efficiency as changes are detected, and by exploiting the considerable redundancy available in the network to operate the nodes on low duty-cycles, one of the primary techniques used today to conserve energy and extend the lifetime of the system to the desired duration.

A WSN is typically designed as a dynamic distributed system, in which complex tasks are performed through the coordinated action of a large number of small devices (nodes). This is especially true for the algorithms devoted to ensuring the efficiency and integrity of the network, which must identify the operating conditions that globally maximize some cost function in a robust and efficient way. Because the information readily available to each node is typically limited to the immediate neighbors, optimization algorithms and decision procedures operate through a series of iterations, in which the network gets progressively closers to a global optimum. Depending on the applications, and certainly in the case of safety-critical systems, it is interesting to study the convergence properties of these algorithms in order to evaluate their effectiveness and establish their correctness.

In past work [11], we have focused on the particular problem of maximizing the sensing coverage of a network under a minimum constraint on lifetime. In particular, we have developed exact off-line solution algorithms for this problem and evaluated the degree of optimality of its distributed implementation, known as wake-up scattering [5]. Later, we have proved local convergence properties of the algorithm (i.e., with respect to small perturbations around an equilibrium points) and global convergence in a few special topology configurations [4]. In this paper, we extend the proof of convergence to the general case.

A. Problem Definition and Background

Given a WSN deployed on a region, we aim at reducing its power consumption (and therefore extend the lifetime) while providing continuous node coverage over a monitored area. To save power, we switch nodes off for a period of time if another node covering the same area is guaranteed to be active. This technique results in a (typically periodic) schedule of the wake-up intervals of the nodes.

An optimal schedule may be computed either centrally and off-line or online by the network itself, in a distributed fashion. Online techniques, preferable for their flexibility and their relative robustness to network topology changes, typically use information from neighboring nodes [5], [12], [6], [2], [1], and pose relevant problems such as algorithm convergence to the solution, how far the solution is from optimal, and how long the transient of the computation lasts. These issues will be covered in the rest of the paper for the *wake–up scattering* problem, proposed in [5].

Briefly, the algorithm computes a periodic schedule over an epoch E, where each node wakes up for only a defined interval of time W. The procedure optimizes the coverage by scattering the wake-up times of neighboring nodes (nodes that can communicate directly over the radio channel), i.e., nodes are scheduled so that they wake up as far in time as possible from neighboring nodes. The rationale behind this approach is the assumption that neighboring nodes are more likely to cover the same area. This is true when the sensing range and the radio range are comparable. While this assumption is clearly an approximation, the technique is extremely simple and relies solely on connectivity, instead of requiring exact position information.

Since the objective of the scattering problem is to find a periodic schedule for the node wake–up times, the problem can be viewed as a deployment task, solved with respect to

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time, over a cyclic set of possible configurations [8], [7], [9]. However, as shown below, there are reasonable situations under which switches in the linear dynamics can happen and the classical analysis on consensus problems cannot be applied to the convergence of the wake-up scattering. Intuitively, the reason of this divergence is the fact that while agents moving on a line are "physically" prevented from overtaking each other, this limitations does not apply to the wake-up times of the nodes. Indeed, as shown below, nodes can change their relative time positions if they do not see each other and a switching behavior can be derived. In [4], local convergence has been established in the general topology case, while global convergence has been provided in special cases by direct analysis of the connectivity of the network. In this paper we extend those results, prove global convergence in the general case and study its application to the coverage problem to show its effectiveness in a practical application.

The paper is organized as follows. In Section II, we provide some the model of the wake-up scattering algorithm, while in Section III we present an analysis on the WSN nodes' visibility instrumental to the stability proof. In Section IV, we show the algorithm convergence in the case of a generic network topology. In Section V, we propose some topology examples and some results related to the coverage problem that clarify the results of the paper and the potential of the algorithm.

II. Model Definition

Consider n nodes $N_1, \dots, N_n$ and let E be the duration of the epoch. We denote by $w_i \in [0, E]$ the wake-up time of node N_i. Let also $\mathcal{V}_i$ be the set of nodes visible from node N_i ($i \notin \mathcal{V}_i$). The wake-up time of node N_i at step k is updated as follows:

$$\begin{aligned} w_i^{k+1} &= (1-\alpha)\, w_i^k + \tfrac{\alpha}{2}\Big(\min_{j \in \mathcal{V}_i}\{w_j^k : w_j^k \geq w_i^k\} + \\ &\quad \max_{j \in \mathcal{V}_i}\{w_j^k : w_j^k \leq w_i^k\}\Big) \mod E. \end{aligned}$$

The initial condition w_i^0, $\forall i$, is randomly chosen providing $w_i^0 < E$ and that $\alpha > 0$ is a design parameter.

To illustrate the formulation, suppose that the epoch E is equivalent to one minute and that the granularity of the wake–up times is the second. In such a case, each w_i corresponds to a position of the second hand, that is invariant to the minute and/or hour chosen. The epoch E defines a ring symmetry, visually equal to the clock dial. For each pair of nodes (N_i, N_j), we define two distances w.r.t. the wake up times w_i and w_j: one, denoted $E \geq \overrightarrow{d}_{i,j} \geq 0$ that goes forward in time, the other, denoted $E \geq \overleftarrow{d}_{i,j} \geq 0$ that goes backward, i.e.,

$$\overrightarrow{d}_{i,j} = \begin{cases} w_j - w_i & \text{if } w_i \leq w_j, \\ w_j - w_i + E & \text{otherwise.} \end{cases}, \tag{1}$$

while $\overleftarrow{d}_{i,j}$ is obtained by equation (1) exchanging w_i with w_j. From the definition above it follows that $\overleftarrow{d}_{i,j} = E - \overrightarrow{d}_{i,j}$. Furthermore, introducing the notation

$$\begin{aligned} \bar{\Delta}^k_{\overrightarrow{d}} &= \tfrac{\alpha}{2}\left(\min_{l \in \mathcal{V}_j}(\overrightarrow{d}^{\,k}_{j,l}) - \min_{l \in \mathcal{V}_i}(\overrightarrow{d}^{\,k}_{i,l})\right) \\ \bar{\Delta}^k_{\overleftarrow{d}} &= \tfrac{\alpha}{2}\left(\min_{l \in \mathcal{V}_j}(\overleftarrow{d}^{\,k}_{j,l}) - \min_{l \in \mathcal{V}_i}(\overleftarrow{d}^{\,k}_{i,l})\right) \end{aligned},$$

we can write the distances update equations as

$$\overrightarrow{d}^{\,k+1}_{i,j} = \overrightarrow{d}^{\,k}_{i,j} + \bar{\Delta}^k_{\overrightarrow{d}} - \bar{\Delta}^k_{\overleftarrow{d}}, \tag{2}$$

$$\overleftarrow{d}^{\,k+1}_{i,j} = \overleftarrow{d}^{\,k}_{i,j} - \bar{\Delta}^k_{\overrightarrow{d}} + \bar{\Delta}^k_{\overleftarrow{d}}. \tag{3}$$

Consider a state vector $\mathbf{x}$ whose entries are the distances $\overrightarrow{d}_{i,l}$, $\forall l \in \mathcal{V}_i$ and $i = 1, \dots, n$. Similarly, let $\mathbf{y}$ be the vector of distances $\overleftarrow{d}_{i,l}$. With the proposed choice of the state variables, the update displacement $\bar{\Delta}^k_{\overrightarrow{d}}$ only depends on the distances in $\mathbf{x}$, and $\bar{\Delta}^k_{\overleftarrow{d}}$ only on the distances in $\mathbf{y}$. By imposing $\mathbf{y} = E\mathbf{1} - \mathbf{x}$, where $\mathbf{1}$ is the column vector of appropriate dimensions with all entries equal to 1, the discrete time evolution of distances is simplified as

$$\mathbf{x}^{k+1} = A\mathbf{x}^k, \tag{4}$$

where A has, at least, one eigenvalue equals to one.

From [4] we know that if two nodes do not see each other, they may overtake each other. This behavior, together with the fact that the updating equations (2) and (3) are nonlinear, makes the overall system dynamics switching. Defining $\eta(k)$ as the switching signal, that takes values $1, \dots, S$, the switching system $\mathbf{x}^{k+1} = A_{\eta(k)}\mathbf{x}^k$ is thus derived, with system matrices $\{A_1, A_2, \dots, A_S\}$. The region of the state space in which the system evolves using a dynamic A_i is a convex polyhedron delimited by a set of subspaces of the type $x_i < x_j$, for appropriate choices of i and j. Hence, in the general case, the number S of linear dynamics is upper bounded by the number of pairs of nodes that do not see each other.

On the other hand, a node cannot overtake any other node that it sees assuming $0 < \alpha < 1$. Therefore, if all nodes see their nearest neighbors, the application of Equation (2) and (3) always produces the same dynamic A_1 and the system evolves with a linear and time–invariant dynamics. As shown in [4], the algorithm in this case asymptotically converges to an equilibrium in which the wake–up times are equally spaced in the epoch E. In the rest of the paper, we will analyze the convergence properties for the general switching system.

III. Visibility Analysis of the WSN

In this section the analysis of a generic WSN is presented in terms of *sub–networks* and *paths*. Without loss of generality, the WSN nodes are supposed ordered, in the sense defined in what follows.

Definition 1: A network Δ with n nodes is called *ordered* if for any pairs of ordered indices $i < j$ implies $w_i < w_j$, $\forall i, j = 1, \dots, n$.

Definition 2: Given an ordered network Δ, a *sub–network* Θ is a set of $n_\theta + 2$ nodes $\{\theta_0, \theta_1, \dots, \theta_{n_\theta}, \theta_{n_\theta+1}\}$, whose wake–up times are ordered $w_{\theta_0} < w_{\theta_1} < \ldots < w_{\theta_{n_\theta}} < w_{\theta_{n_\theta+1}}$

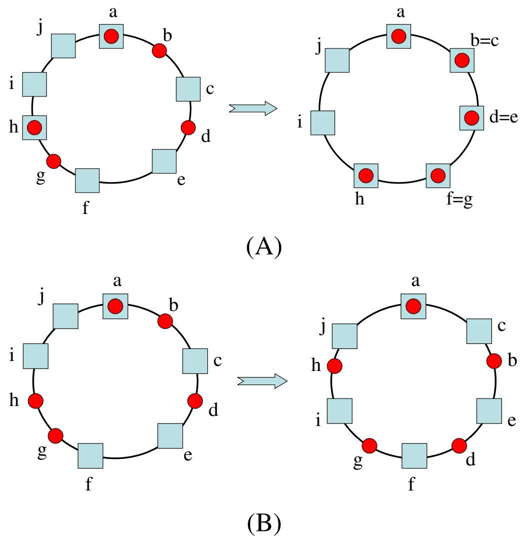


Fig. 1. Nearest neighbor visibility and proximity of the connected sub–networks topology. Nearest neighbors are the closest nodes in clockwise and counter–clockwise direction, i.e. the nearest neighbor to c are d and b respectively. Visibility is depicted with squares and circles, hence the nearest *visible* nodes from c are a and e, respectively. The pictures on the left depict the initial configuration of the networks, while on the right the corresponding equilibrium is shown. The circumferences are of length E. (A) $\{b, d, g\} \not\in \mathcal{V}_{c,e,f}$ and viceversa. (B) $\{b, d, g, h\} \not\in \mathcal{V}_{c,e,f,i,j}$ and viceversa.

and such that node θ_i sees (at least) nodes θ_{i-1} and θ_{i+1}. Nodes θ_0 and $\theta_{n_\theta+1}$ are defined as the *end–points* of the sub–network.

The previous definitions are instrumental for the subsequent definition of partial visibility network by means of sub–networks.

Definition 3: Given an ordered network Δ and two sub–networks Θ and Σ whose end–points are coincident and equal to $\theta_0 \equiv \sigma_0 \equiv \nu_i$ and $\theta_{n_\theta+1} \equiv \sigma_{n_\sigma+1} \equiv \nu_e$ respectively, Θ and Σ are named *connected sub–networks* $\Theta \leftrightsquigarrow \Sigma$ if any element of Θ does not see any element of Σ, except the end–points ν_i and ν_e.

Example 1: The network of figure 1-(A) has $\Theta = \{a, b, d, g, h\}$ and $\Sigma = \{a, c, e, f, h\}$, with $\Theta \leftrightsquigarrow \Sigma$. The network of figure 1-(B) has $\Theta = \{a, c, e, f, i, j, a\}$ and $\Sigma = \{a, b, d, g, h, a\}$, with, again, $\Theta \leftrightsquigarrow \Sigma$.

Definition 4: Given an ordered network Δ, a *path* π^i of length l_i is a subset of ordered nodes of Δ, i.e., $\{\delta_{\pi^i_1}, \delta_{\pi^i_2}, \ldots, \delta_{\pi^i_{l_i}}\}$, such that the following properties hold:

1) *Proximity*: $\delta_{\pi^i_j}$ sees at least $\delta_{\pi^i_{j+1}}$;
2) *Recurrence*: $\delta_{\pi^i_{l_i}} = \delta_{\pi^i_1}$;
3) *Size*: $\sum_{j=1}^{l_i-1} \overrightarrow{d}_{\delta_{\pi^i_j}, \delta_{\pi^i_{j+1}}} = E$.

The notation $\delta_{\pi^i_j}$ should be intended as "the j–th node of π^i belonging to the ordered WSN Δ".

In practice, a path $\bar{\pi}^i$ *is* a network with nearest neighbor visibility.

Definition 5: A *sequence* is a subset of path nodes that verifies only the proximity property.

Definition 6: Two paths π^i and π^q are *equivalent*, i.e., $\pi^i \sim \pi^q$, if π^q can be obtained by π^i by a node permutation that preserves the order.

As an example of Definition 6, $\pi^i = \{\delta_{\pi^i_1}, \delta_{\pi^i_2}, \ldots, \delta_{\pi^i_{l_i}}\}$ is equivalent to $\pi^q = \{\delta_{\pi^i_j}, \ldots, \delta_{\pi^i_{l_i}}, \delta_{\pi^i_2}, \ldots, \delta_{\pi^i_j}\}$.

Definition 7: Given an ordered network Δ, let $\mathcal{P}$ be the set of all possible paths. The equivalence class w.r.t. the equivalence relation "$\sim$" is denoted by $\pi^{i\star} = \{\pi^q \in \mathcal{P} | \pi^q \sim \pi^i\}$. Furthermore, let $\bar{\pi}^i$ be the representative path of the equivalence class of $\pi^{i\star}$, i.e., a path randomly selected from $\pi^{i\star}$.

Definition 8: Let $\bar{\mathcal{P}}$ be the set of path representatives (one for each equivalence class) such that:

1) $\forall \bar{\pi}^i, \bar{\pi}^j \in \bar{\mathcal{P}} \Rightarrow l_i \geq l_j$ if $i < j$;
2) $\forall \bar{\pi}^i, \bar{\pi}^j \in \bar{\mathcal{P}} \Rightarrow \bar{\pi}^i \not\subset \bar{\pi}^j$.

In light of the previous definitions, we are now able to make a link between the paths and the connected sub–networks. Indeed, if $\bar{\pi}^i \cap \bar{\pi}^j = \{\delta_k, \ldots, \delta_{k+m}\}$, with $m \geq 1$ and $\delta_k, \ldots, \delta_{k+m}$ is a sequence belonging to both $\bar{\pi}^i$ and $\bar{\pi}^j$, then there exists two connected sub–networks, $\Sigma \in \bar{\pi}^i$ and $\Theta \in \bar{\pi}^j$, having start–point δ_{k+m} and end–point δ_k. Notice that the number of connected sub–networks for $\bar{\pi}^i$ and $\bar{\pi}^j$ is then equal to the number of sequences belonging to both $\bar{\pi}^i$ and $\bar{\pi}^j$.

Example 2: Let us consider the case of connected sub–networks, represented in figure 1-(A). The set $\bar{\mathcal{P}}$ comprises 2 strings, i.e.,

$$\begin{cases} \bar{\pi}^1 = \{a, c, e, f, h, i, j, a\} \\ \bar{\pi}^2 = \{a, b, d, g, h, i, j, a\} \end{cases},$$

that comprises the connected sub–networks of Example 1.

Example 3: Let us consider the case of connected sub–networks, represented in figure 1-(B). The set $\bar{\mathcal{P}}$ has 2 strings, i.e.,

$$\begin{cases} \bar{\pi}^1 = \{a, c, e, f, i, j, a\} \\ \bar{\pi}^2 = \{a, b, d, g, h, a\} \end{cases},$$

the connected sub–networks of Example 1.

Example 4: Consider two paths of the same length given by

$$\begin{cases} \bar{\pi}^1 = \{a, b, c, d, e, f, g, a\} \\ \bar{\pi}^2 = \{a, h, i, d, e, j, k, a\} \end{cases},$$

then the two paths define two pairs of connected sub–networks.

IV. Stability Analysis

Before going into details, the following Lemma, presented in [4] and describing the convergence of the scattering algorithm when no switching occurs, is reported.

Lemma 1: Given the system $\mathbf{x}^{k+1} = A\mathbf{x}^k$ and $\mathbf{x}^0 \in \mathcal{S}_E^{n_x}$, with $\mathcal{S}_E^{n_x} = \{\mathbf{x} \in \mathbb{R}^{n_x} | 0 \leq x_i \leq E\}$ and n_x the dimension of $\mathbf{x}$, the following statements hold true:

- $\mathbf{x}^k \in \mathcal{S}_E^{n_x}$ $\forall k > 0$;
- the system is stable;
- the equilibrium points $\bar{\mathbf{x}}$ belong to a linear subspace defined by the $m \geq 1$ eigenvectors $\mathbf{v}_i$ associated to the m eigenvalues $\lambda_i = 1$.

The main result of this Lemma, that will be used in the sequel, is related to the sign invariance of distances between nodes that see each other, i.e., nodes that see each other never overtake each other, regardless of the switchings that may occur (see [4]).

A. Global Stability for Connected Sub–Networks

Consider an ordered network with two connected sub–networks Θ and Σ with end–points ν_i and ν_e. Let $\theta_{-1} \equiv \sigma_{-1} \equiv \nu_i'$ be the nearest node to node ν_i in the counterclockwise direction (ν_e' be the nearest node to node ν_e in clockwise direction). Let ξ be the generic nearest node to ν_i in the clockwise direction (either θ_1 or σ_1).

The update equations for Θ can be written as

$$\begin{array}{rcl}\overrightarrow{d}^{k+1}_{\theta_i,\theta_{i+1}} & = & \overrightarrow{d}^{k}_{\theta_i,\theta_{i+1}} + \frac{\alpha}{2}(\overrightarrow{d}^{k}_{\theta_{i+1},\theta_{i+2}} - \overrightarrow{d}^{k}_{\theta_i,\theta_{i+1}}) - \\ & & \frac{\alpha}{2}(\overrightarrow{d}^{k}_{\theta_i,\theta_{i+1}} - \overrightarrow{d}^{k}_{\theta_{i-1},\theta_i}),\end{array} \quad (5)$$

$\forall i = 1, \ldots, n_\theta - 1$, where $\theta_{i+2} = \nu_e$ for $i = n_\theta - 1$. For $i = 0$ we have

$$\overrightarrow{d}^{k+1}_{\nu_i,\theta_1} = \overrightarrow{d}^{k}_{\nu_i,\theta_1} + \frac{\alpha}{2}(\overrightarrow{d}^{k}_{\theta_1,\theta_2} - \overrightarrow{d}^{k}_{\nu_i,\xi}) - \frac{\alpha}{2}(\overrightarrow{d}^{k}_{\nu_i,\theta_1} - \overrightarrow{d}^{k}_{\nu_i',\nu_i}). \quad (6)$$

Substituting the index θ_i with σ_i, the update equations for the Σ sub–network is obtained.

Since node θ_i, $\forall i$, does not see any node in Σ, it can overtake any node in Σ. Hence, the following Lemma holds.

Lemma 2: Given an ordered network and two connected sub–networks Θ and Σ, whose end–points are ν_i and ν_e, the scattering dynamic matrix A in (4) switches if and only if $\overrightarrow{d}_{\theta_1,\sigma_1}$ or $\overrightarrow{d}_{\theta_{n_\theta},\sigma_{n_\sigma}}$ changes sign over time.

Since the switchings are then state dependent, it is necessary to study the dynamics of the distances between nodes that do not see each other, i.e., $\overrightarrow{d}_{\theta_i,\sigma_i}$. The rationale of the analysis that follows stems from the subsequent relations of general validity

$$\overrightarrow{d}_{\nu_i,\theta_i} = \overrightarrow{d}_{\nu_i,\theta_1} + \sum_{j=1}^{i-1} \overrightarrow{d}_{\theta_i,\theta_{i+1}} \quad (7)$$

$$\overrightarrow{d}_{\theta_i,\sigma_i} = \overrightarrow{d}_{\nu_i,\sigma_i} - \overrightarrow{d}_{\nu_i,\theta_i}, \quad (8)$$

$$\overrightarrow{d}_{\theta_{i+1},\sigma_{i+1}} - \overrightarrow{d}_{\theta_i,\sigma_i} = \overrightarrow{d}_{\sigma_i,\sigma_{i+1}} - \overrightarrow{d}_{\theta_i,\theta_{i+1}}, \quad (9)$$

$$\overrightarrow{d}_{\theta_{n_\theta},\sigma_{n_\sigma}} = \overrightarrow{d}_{\theta_{n_\theta},\nu_e} - \overrightarrow{d}_{\sigma_{n_\sigma},\nu_e}. \quad (10)$$

Equations (7) and (8) follow trivially from the ring topology and the fact that $\nu_i \in \Theta$ and $\nu_i \in \Sigma$. Equation (9) is obtained by substituting equation (8), for i and $i+1$, in the left hand side terms of the equation (9). Equation (10) follows from equation (8) computed for counterclockwise distances and then substituting the epoch E invariance property.

Remark 1: The analysis carried out in this section is presented for clockwise distances. Nevertheless, similar results can be obtained using counterclockwise distances.

Consider the case in which $n_\theta > n_\sigma$ (as in the figure 1-(B)). Substituting the update equations $\overrightarrow{d}^{k+1}_{\nu_i,\theta_i}$ and $\overrightarrow{d}^{k+1}_{\nu_i,\sigma_i}$ into equation (5) and using the equations (7), (8) and (9), one gets

$$\overrightarrow{d}^{k+1}_{\theta_i,\sigma_i} = (1-\alpha)\overrightarrow{d}^{k}_{\theta_i,\sigma_i} + \frac{\alpha}{2}\overrightarrow{d}^{k}_{\theta_{i+1},\sigma_{i+1}} + \frac{\alpha}{2}\overrightarrow{d}^{k}_{\theta_{i-1},\sigma_{i-1}}. \quad (11)$$

Notice that for $i = 1$, $\overrightarrow{d}_{\theta_{i-1},\sigma_{i-1}} = \overrightarrow{d}_{\nu_i,\nu_i} = 0$. Furthermore, for $i = n_\sigma$, $\overrightarrow{d}_{\theta_{i+1},\sigma_{i+1}} = \overrightarrow{d}_{\theta_{i+1},\nu_e}$. Therefore, let $\mathbf{z} = [\overrightarrow{d}_{\theta_1,\sigma_1}, \ldots, \overrightarrow{d}_{\theta_{n_\sigma},\sigma_{n_\sigma}}]^T$, be the state vector of distances between nodes that do not see each other till n_σ, and consider its linear dynamics $\mathbf{z}^{k+1} = A_{cn}\mathbf{z}^k + \mathbf{c}^k$, where $\mathbf{c}^k = [0, \ldots, 0, \alpha/2\overrightarrow{d}^{k}_{\theta_{i+1},\nu_e}]^T$. By the update equations (11) follows trivially that A_{cn} is a tridiagonal Toeplitz matrix that has n_σ distinct eigenvalues λ_i. Furthermore, for $0 < \alpha < 1$, A_{cn} turns to be a Schur matrix which is also non negative (recall (11)). The existence of an equilibrium point for $\mathbf{z}$ is proved in what follows.

Lemma 3: The state space vector $\mathbf{z}$ converges to an equilibrium $\bar{\mathbf{z}} > 0$.

Proof. Since the term $\mathbf{c}^k \geq 0$, $\forall k$, and A_{cn} is a non negative Schur matrix, it follows that for $k > \tilde{k} > 0$, $\mathbf{z}^k \geq 0$. Hence, the node closer to ν_i belongs to Θ for $k > \tilde{k}$. Writing the system for counterclockwise distances, it follows similarly that the node closer to ν_e belongs again to Θ for $k > \tilde{k}$. From Lemma 2 no switchings will ever occur and then, from Lemma 1, $\overrightarrow{d}^{k}_{\theta_{i+1},\nu_e}$ is constant for $k > \tilde{k}$. It turns out that the equilibrium point is given by $\bar{\mathbf{z}} = (I_{n_\sigma} - A_{cn})^{-1}\mathbf{c}$, where I_{n_σ} is the identity matrix of appropriate dimensions. Using results from linear algebra reported in [3], we have that $(I_{n_\sigma} - A_{cn})^{-1}$ is a positive matrix, hence $\bar{\mathbf{z}} > 0$. ■

Lemma 4: Given an ordered network Δ with two connected sub–networks Θ and Σ, whose end–points are ν_i and ν_e, and with $n_\theta > n_\sigma$, the dynamics of node distances in Σ is given by a linear system whose dynamic matrix A_σ is doubly stochastic and whose input $\mathbf{c}^k_\sigma$ is bounded.

Proof. Consider the state vector defined by the node distances of Σ, i.e., $\mathbf{z}_\sigma = [\overrightarrow{d}_{\nu_i,\sigma_1}, \ldots, \overrightarrow{d}_{\sigma_{n_\sigma},\nu_e}]^T$. From Lemma 3, the nodes ν_i and ν_e are not influenced by the nodes in Σ after the transient. Therefore, from (6), $\mathbf{z}^{k+1}_\sigma = A_\sigma\mathbf{z}^k_\sigma + \mathbf{c}^k_\sigma$, where A_σ is a Schur matrix that is tridiagonal, doubly stochastic and non negative by construction. The input vector is instead given by $\mathbf{c}^k_\sigma = \alpha/2[(\overrightarrow{d}^{k}_{\nu_i',\nu_i} - \overrightarrow{d}^{k}_{\nu_i,\theta_1}), 0, \ldots, 0, (\overrightarrow{d}^{k}_{\nu_e,\nu_e'} - \overrightarrow{d}^{k}_{\theta_{n_\theta},\nu_e})]^T$. ■

Therefore, recalling the definition of paths $\bar{\pi}^i$, the following summarizing Theorem holds.

Theorem 1: Consider a network with exactly two paths, $\bar{\pi}^1$ and $\bar{\pi}^2$ with $l_1 > l_2$, and such that $\bar{\pi}^1 \cap \bar{\pi}^2 = \{\delta_k, \ldots, \delta_{k+m-1}\}$ is a sequence of length m. Let $\Theta \in \bar{\pi}^1$ and $\Sigma \in \bar{\pi}^2$ be the two connected sub–networks thus defined. The equilibrium distances of the nodes in $\bar{\pi}^1$ is E/l_1, while the equilibrium distances of the nodes in $\bar{\pi}^2 \backslash (\bar{\pi}^1 \cap \bar{\pi}^2)$ is given by

$$\frac{E(l_1 - m)}{l_1(l_2 - m)}.$$

Proof. In view of Lemma 2, the switchings are determined by the sign of the first and the last element of $\mathbf{z}$. Since $\bar{\mathbf{z}} > 0$ (see Lemma 3), the dynamics of the network is eventually governed by the $l_1 = n - n_\sigma$ nodes with nearest neighbor visibility of $\bar{\pi}^1$, for which the size property of Definition 4 holds. Since for nearest neighbor visibility the dynamic matrix is a doubly stochastic matrix (see [4], [10]), each distance converges to the mean of all distances, hence E/l_1.

The equilibrium of the node distances of the path $\bar{\pi}^2$ is unknown only for the connected sub–network $\Sigma \in \bar{\pi}^2$. Since the equilibrium of nodes ν_i and ν_e is given by the path $\bar{\pi}^1$ and in view of Lemma 4, $\mathbf{c}_\sigma^k \to 0$ for $k \to +\infty$ and the distances between the nodes in Σ will be all equal at the equilibrium. Since the overall distance between the end–points of Σ is given by $E(n_\theta + 1)/(n - n_\sigma)$, the proof follows. ■

Let us summarize the previous result in matrix terms. With reference to the longest path $\bar{\pi}^1$, the dynamic of the distances between nearest neighbor nodes $\mathbf{x}_{\bar{\pi}^1}$ can be expressed by a single matrix $A_{\bar{\pi}^1} \in \mathbb{R}^{(l_1-1)\times(l_1-1)}$. The dynamic of the distances of the network $\bar{\pi}^1 \cup \bar{\pi}^2$ is determined by the switching matrix set $\mathcal{A}_{\bar{\pi}^1\cup\bar{\pi}^2}$ and it is described by the vector $\mathbf{x}_{\bar{\pi}^1\cup\bar{\pi}^2} = [\mathbf{x}_{\bar{\pi}^1}^T, \mathbf{x}_{\bar{\pi}^2/(\bar{\pi}^1\cap\bar{\pi}^2)}^T]^T$. For instance, with reference to Example 2 (figure 1-(A)), $\mathbf{x}_{\bar{\pi}^1} = [\overrightarrow{d}_{a,c}, \overrightarrow{d}_{c,e}, \overrightarrow{d}_{e,f}, \overrightarrow{d}_{f,h}, \overrightarrow{d}_{h,i}, \overrightarrow{d}_{i,j}, \overrightarrow{d}_{j,a}]^T$ and $\mathbf{x}_{\bar{\pi}^2/(\bar{\pi}^1\cap\bar{\pi}^2)} = [\overrightarrow{d}_{a,b}, \overrightarrow{d}_{b,d}, \overrightarrow{d}_{d,g}, \overrightarrow{d}_{g,h}]^T$. By means of Theorem 1, the matrix governing the dynamic of the distances after the transient is given by

$$A_{\bar{\pi}^1\cup\bar{\pi}^2} = \begin{bmatrix} A_{\bar{\pi}^1} & 0 \\ B_{\bar{\pi}^1\cup\bar{\pi}^2} & A_{\bar{\pi}^2} \end{bmatrix} \in \mathcal{A}_{\bar{\pi}^1\cup\bar{\pi}^2},$$

where $A_{\bar{\pi}^1}$ is a doubly stochastic circulant matrix and $A_{\bar{\pi}^2}$ is a doubly stochastic tridiagonal matrix. Of course, if $\bar{\pi}^1\cap\bar{\pi}^2 = \emptyset$, $B_{\bar{\pi}^1\cup\bar{\pi}^2} = 0$ and $\mathcal{A}_{\bar{\pi}^1\cup\bar{\pi}^2}$ contains only one block diagonal matrix.

It is worthwhile to note that if there are two (or more) sequences, $\delta_k, \ldots, \delta_{k+n}$ and $\delta_l, \ldots, \delta_{l+n}$ that belongs to both $\bar{\pi}^i$ and $\bar{\pi}^j$, then there are two (or more) pairs of sub–networks, one having start–point δ_{k+n} and end–point δ_l and the other one having start–point δ_{l+n} and end–point δ_k. However, the Theorem 1 is still valid for each pair of connected sub–networks. Moreover, notice that the set of all the paths in $\bar{\mathcal{P}}$, represent all the possible dynamics for the scattering algorithm, i.e., determines all the possible dynamic matrices $A_{\eta(k)}$. For instance, for nearest neighbor visibility, $\bar{\mathcal{P}}$ contains only one path.

Remark 2: Since $A_{cn} \geq 0$ of Lemma 3 is Schur, we are able to discuss the switchings of the system with respect to the parameters choice and to the initial configuration of the network, i.e., $\mathbf{z}(0)$:

1) If $\mathbf{z}(0) \geq 0$, no switching occurs;
2) For a generic $\mathbf{z}(0)$, if $\alpha \leq 1/(1-\cos(i\pi/(n_\theta+1))\ \forall i$, then $0 \leq \lambda_i < 1$ and a limited number of switchings exists;
3) If $\alpha > 1/(1-\cos(i\pi/(n_\theta+1))$, due to presence of at least one $\lambda_i < 0$, an infinite number of switchings cannot be ruled out since the linear system presents an oscillating behavior of the state variables in $\mathbf{z}$. Nevertheless, since the oscillations are damped, the system will practically eventually reach the steady state value.

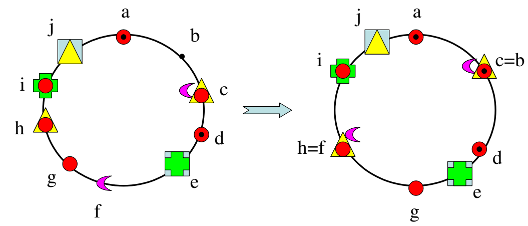


Fig. 2. Nearest neighbor visibility and proximity for a generic network topology. Visibility is depicted with different symbols. Notice that if a node has m associated symbols, it means that m different nodes can be the nearest for some instants. The pictures on the left depict the initial configuration of the network, while on the right the equilibrium is shown. The circumferences are of length E.

Remark 3: In the case of $n_\theta = n_\sigma$ (as in the figure 1-(A)), Lemma 3 is still valid with $\mathbf{c}^k = 0$, $\forall k$. Hence, $\mathbf{z} = 0$ is the unique equilibrium point. Similarly, Theorem 1 holds also in this case with equilibrium distances equal for the two paths, i.e., the equilibrium *is* the switching surface between two different dynamics.

Remark 4: Recalling equation (11), the analysis still works if $\nu_i \equiv \nu_e$ (depicted in figure 1-(B)).

B. Global Stability for a Generic Network Topology

To prove the convergence in the general case, we will make use of the definition of paths $\bar{\pi}^i$ to construct all the possible sub–networks for a generic network topology (see figure 2).

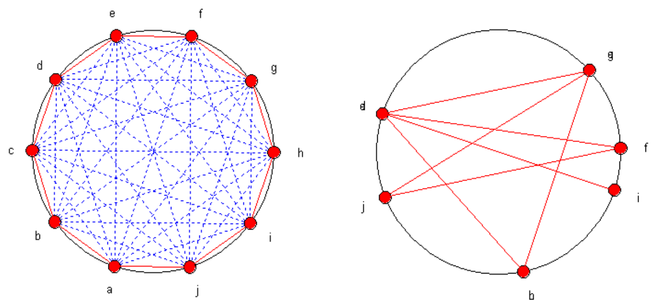
Example 5: Let us consider the generic network topology depicted in figure 2. The $\bar{\mathcal{P}}$ has 6 strings, i.e.,

$$\left\{\begin{array}{lcl} \bar{\pi}^1 & = & \{a,c,d,g,h,i,a\} \\ \bar{\pi}^2 & = & \{a,b,d,g,h,i,a\} \\ \bar{\pi}^3 & = & \{c,d,g,h,j,c\} \\ \bar{\pi}^4 & = & \{e,j,e\} \\ \bar{\pi}^5 & = & \{e,i,e\} \\ \bar{\pi}^6 & = & \{c,f,c\} \end{array}\right. .$$

The final result of the paper on the global convergence of the distributed wake–up scattering algorithm is presented in the sequel.

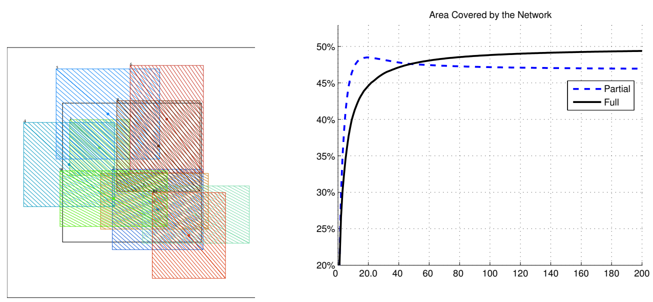
Theorem 2: Given a generic network, the wake–up scattering algorithm asymptotically converges towards an equilibrium where the node distances are related to the path length l_1.

For space limits, we present here only a sketch of the proof, which is based on an extension of Lemma 3 and Lemma 4. Consider two paths that simultaneously generate connected sub–networks with $\bar{\pi}^1$, i.e., $\bar{\pi}^1\cap\bar{\pi}^q\cap\bar{\pi}^p \neq \emptyset$. As in the previous case, it is again possible to define the state vector of all the distances between nodes that do not see each other. Since the nodes of the connected sub–networks may be also inter–connected, a set of switching dynamic matrices are then obtained. Nevertheless, it can be shown that Lemma 3 can be extended to this case. Generalizing to an arbitrary number of paths sharing a sequence of nodes with $\bar{\pi}^1$, the stability for the nodes in $\bar{\pi}^1$, with distance equilibrium determined by E/l_1 (Theorem 1), is derived. The stability of the nodes



(A) (B)

Fig. 3. Steady state wake-up times. (A) Complete visibility. (B) Partial visibility.



(A) (B)

Fig. 4. The coverage scenario. (A) Spatial distribution of the nodes, (B) Evolution of the ratio between covered area and coverable area.

for all the paths $\bar{\pi}^j$ with $j > 1$ follow from the generalized Lemma 4.

V. Topology Examples

In this section, we provide some numerical evidence of the effectiveness of the approach. We consider a very simple deployment consisting of 10 nodes. The epoch for the schedule equal to 5 time units and a wake-up interval for the nodes equal to 1. Therefore, each node is awake for 20% of the total time. In the simulations presented $\alpha = 0.5$ and in the initial configuration all the nodes have the same wake–up time.

In figure 3.(A), the final configuration reached by the network wake–up times in the case of nearest neighbor visibility is depicted. Node names are represented with letters. Solid lines connects nearest visible nodes, while dashed lines connect visible but not nearest nodes. Notice that the network presents full visibility among nodes. In the case of partial visibility, the steady state wake–up times are depicted in figure 3.(B). In this case, some nodes converge in the same position, i.e., they are switched on and off in the same time instant.

In order to show the performance of the wake–up scattering algorithm for the coverage problem, we consider a rectangular sensing range for the nodes. The nodes are randomly distributed over a 500×500 bi-dimensional area. The deployment in the environment is shown in Figure 4.(A). Several regions of the considered arena are covered by multiple nodes. Therefore, a good schedule is one where the wake-up times of nodes sharing "large" areas are far apart. Using the algorithm presented in [11], we come up with an optimal schedule, where an average of 52.94% of the "coverable" area (i.e., the area actually within the sensing range of the nodes) is actually covered. The application of the wake-up scattering algorithm, assuming alternatively complete and partial visibility between the nodes, produces the result shown in Figure 4.(B). The attained relative coverage over 100 experiments is 47.3%, with a deviation from the optimal solution of less than 10% of the optimal coverage, in the full visibility case.

VI. Conclusions

In this paper, we have presented convergence results of a distributed algorithm used for maximizing the lifetime of a WSN. We have focused our attention on an algorithm recently proposed in the literature, showing how its convergence can be cast into a stability problem for a linear switching system. We have proved the stability of the distributed algorithm in the general case, starting from the analysis of specific topologies of the WSN. An application of the scattering algorithm to the coverage problem has been presented to prove the relevance of the approach in a practical problem.

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