Course on Model Predictive Control
Part III – Stability and robustness

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Outline

1. Nominal stability analysis
   - Preliminaries on stability analysis and Lyapunov functions
   - Closed loop description
   - Stability results
2. Nominal (inherent) robustness
   - Perturbed closed-loop system
   - Robust stability and recursive feasibility
3. Suboptimal MPC: stability and robustness
4. Robust MPC design
   - Min-max
   - Tube-based robust MPC
5. Output feedback MPC
   - Stability analysis
   - Offset-free MPC analysis and design

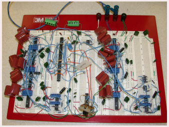
Some preliminary definitions

Discrete-time system

- Consider **general nonlinear** discrete-time systems:

$$x^+ = f(x, u)$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ **continuous**



- Let $\phi(k; x, \mathbf{u})$ be the **solution** of $x^+ = f(x, u)$ **at time** k for **initial state** $x(0) = x$ and **control sequence** $\mathbf{u} = \{u(0), u(1), \ldots\}$
- **Given a state-feedback** law $u = \kappa(x)$, obtain a **closed-loop**

$x^+ = f(x, \kappa(x))$ denote again the solution as $\phi(k; x)$

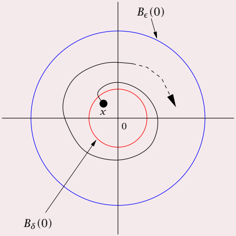
Equilibrium and positive invariance

- A **point** x^* is an **equilibrium point** of $x^+ = f(x, \kappa(x))$ if $x(0) = x^*$ implies that $x(k) = \phi(k; x^*) = x^*$ **for all** $k \geq 0$
- A **set** A is **positively invariant** for $x^+ = f(x, \kappa(x))$ if $x \in A$ implies that $x^+ = f(x, \kappa(x)) \in A$

Stability and asymptotic stability

Stability and attractivity of the origin

- Given a (**closed-loop**) system $x^+ = f(x)$, with the **origin as equilibrium**, i.e. $f(0) = 0$
- The **origin is locally stable** if for **every $\epsilon > 0$**, there **exists $\delta > 0$** such that $|x| < \delta$ implies $|\phi(k;x)| < \epsilon$
- The **origin is globally attractive** if $\lim_{k\to\infty} |\phi(k;x)| = 0$ for any $x \in \mathbb{R}^n$



Global asymptotic stability and exponential stability

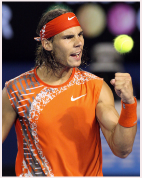
- The **origin is globally**
  - **asymptotically stable** (GAS) if it is **locally stable** and **globally attractive**
  - **exponentially stable** (GES) if there exist $c > 0$ and $\gamma \in (0,1)$ such that:

$$|\phi(k;x)| \le c|x|\gamma^k \quad \text{for all } k \ge 0$$

Asymptotic stability for constrained systems

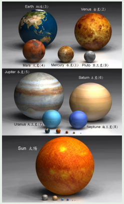
GAS for constrained system

- Let $\mathbb{X}$ be **positively invariant** for $x^+ = f(x)$
- The origin is
  - **locally stable** in $\mathbb{X}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $x \in \mathbb{X} \cap \delta\mathbb{B}$ there holds $|\phi(k; x)| < \epsilon$ for all $k \geq 0$
  - **attractive** if for every $x \in \mathbb{X}$ there holds $\lim_{k\to\infty} |\phi(k; x)| = 0$
  - **asymptotically stable** in $\mathbb{X}$ if it is **locally stable** and **attractive**
- $\mathbb{X}$ is called **region (or domain) of attraction** for the origin



Comparison function

- A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of **class $\mathcal{K}$** if it is **continuous**, $\sigma(0) = 0$ and **strictly increasing** ($\mathcal{K}_\infty$ if unbounded)
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$ is of **class $\mathcal{KL}$** if for **each** $t \in \mathbb{N}$, $\beta(\cdot, t)$ is a $\mathcal{K}$ function, and **for each** $s \in \mathbb{R}_{\geq 0}$, $\lim_{t\to\infty} \beta(s, t) = 0$
- GAS is **equivalent** to $|\phi(k; x)| \leq \beta(|x|, k)$ for all $k \geq 0$, $\beta(\cdot) \in \mathcal{KL}$



Lyapunov functions and asymptotic stability

General definition

- A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a **Lyapunov function** for $x^+ = f(x)$ if there **exist** $\mathcal{K}_\infty$ **functions** $\alpha_1, \alpha_2, \alpha_3$ such that **for all** $x \in \mathbb{R}^n$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

$$V(f(x)) - V(x) \leq -\alpha_3(|x|)$$

- V **decreases** during the evolution of the system

Lyapunov functions and GAS

If $V(\cdot)$ is a **Lyapunov function** for $x^+ = f(x)$, the origin is **globally asymptotically stable**

Lyapunov functions and stability for constrained systems

Asymptotic stability

Then, the origin is **asymptotically stable** in $\mathbb{X}$ if:

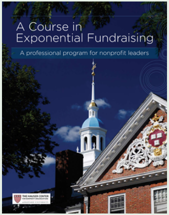
- $\mathbb{X}$ is **positively invariant** for $x^+ = f(x)$
- $V(\cdot)$ is a **Lyapunov function** for $x^+ = f(x)$

Exponential Lyapunov function and stability

The origin of $x^+ = f(x)$ is **exponentially stable** in $\mathbb{X}$ if

- $\mathbb{X}$ is **positively invariant** for $x^+ = f(x)$
- There exist $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and **positive constants** a, a_1, a_2, a_3:

$$a_1 |x|^a \leq V(x) \leq a_2 |x|^a$$
$$V(f(x)) - V(x) \leq -a_3 |x|^a$$



Prototype MPC problem

- Given **current state** $x(0) = x$, solve for the input sequence $\mathbf{u} = \{u(0;x), u(1;x), \dots, u(N-1;x)\}$

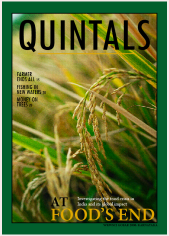
$$\mathbb{P}_N(x): \qquad \min_{\mathbf{u}} V_N(x,\mathbf{u}) \qquad \text{s.t.}$$

$$x^+ = Ax + Bu$$

$$x(j) \in \mathbb{X} \qquad \text{for all } j = 0,\dots,N-1$$

$$u(j) \in \mathbb{U} \qquad \text{for all } j = 0,\dots,N-1$$

$$x(N) \in \mathbb{X}_f$$



- **Cost function**:

$$V_N(x,\mathbf{u}) = \sum_{j=0}^{N-1} \ell(x(j),u(j)) + V_f(x(N)), \qquad \ell(x,u) = x'Qx + u'Ru$$

- **Terminal** cost: $V_f(x) = x'Px$

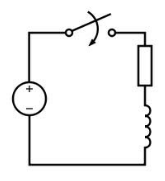
Closed-loop system and basic path for stability

Closed-loop system

- Given the **optimal solution sequence** $\mathbf{u}^0(x)$, function of current state x, denote the **implicit MPC control** law

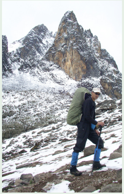
$$\kappa_N(x) = u^0(0; x)$$

- Closed-loop system: $x^+ = Ax + B\kappa_N(x)$
- Notice that $\kappa_N : \mathcal{X}_N \to \mathbb{U}$ is **not linear**



Basic route to prove stability

- Show that $V_N^0(\cdot)$ is a **Lyapunov function** for $x^+ = f(x) = Ax + \kappa_N(x)$
- Show that the **feasibility set**, $\mathcal{X}_N$, is **positively invariant**
- (**Control invariance** of $\mathbb{X}_f$) For every $x \in \mathbb{X}_f$, there exists $u \in \mathbb{U}$: $x^+ = Ax + Bu \in \mathbb{X}_f$ $\qquad V_f(x^+) - V_f(x) \leq -\ell(x, u)$



Stability proof

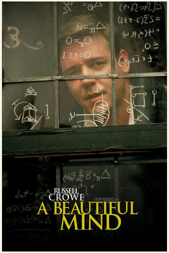
Lemma. Optimal cost decrease

For all $x \in \mathcal{X}_N$, there holds: $V_N^0(Ax + B\kappa_N(x)) - V_N^0(x) \le -\ell(x, \kappa_N(x))$

Proof

- Consider the **optimal input and state sequences**
  $\mathbf{u}^0(x) = \{u^0(0;x), u^0(1;x), \ldots, u^0(N-1;x)\}$
  $\mathbf{x}^0(x) = \{x^0(0), x^0(1), \ldots, x^0(N)\}$
- At **next time**, given $x^+ = Ax + B\kappa_N(x)$, consider a **candidate sequence** $\tilde{\mathbf{u}} := \{u^0(1;x), \ldots, u^0(N-1;x), u(N)\}$
- **Choose** $u(N) \in \mathbb{U}$ such that $x(N+1) = Ax^0(N;x) + Bu(N) \in \mathbb{X}_f$ and $V_f(x(N+1)) + \ell(x(N), u(N)) \le V_f(x^0(N))$
- $\tilde{\mathbf{u}}$ is **feasible** and $V_N(x^+, \tilde{\mathbf{u}}) \le V_N^0(x) - \ell(x, \kappa_N(x))$
- But **not optimal** for $\mathbb{P}_N(x^+)$. Thus:

$$V_N^0(x^+) \le V_N(x^+, \tilde{\mathbf{u}}) \le V_N^0(x) - \ell(x, \kappa_N(x))$$



Examples of linear MPC: the origin as terminal set

Simple idea

- (No) **Terminal cost**: $V_f(x) = 0$
- **Terminal set**: $\mathbb{X}_f = \{0\}$

Drawbacks

- The **feasibility set** $\mathcal{X}_N$ may be **small** because one needs to **reach the origin** in N steps (with **constrained input** $u \in \mathbb{U}$)
- **Closed-loop evolution** of $x^+ = Ax + B\kappa_N(x)$ and **open-loop** trajectory $\{x^0(0), x^0(1), \ldots, x^0(N-1), 0\}$ may be **very different**

Examples of linear MPC: Rawlings and Muske [1993]

Open-loop stable systems

- **Terminal cost**: $V_f(x) := x'Px$ with P **solution** to the **Lyapunov equation**:

$$P = A'PA + Q \qquad \text{notice that: } P = \sum_{j=0}^{\infty} (A^j)' Q A^j$$

- **Terminal set** $\mathbb{X}_f = \mathbb{X}$

Open-loop unstable systems

- Perform **Schur decomposition**: $A = \begin{bmatrix} S_s & S_u \end{bmatrix} \begin{bmatrix} A_s & A_{su} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} S_s' \\ S_u' \end{bmatrix}$
- Solve **reduced** Lyapunov equation: $\Pi = A_s' \Pi A_s + S_s' Q S_s$
- **Terminal cost**: $V_f(x) = x'Px$ with $P = S_s' \Pi S_s$
- **Terminal set**: $\mathbb{X}_f = \{x \in \mathbb{X} \mid S_u' x = 0\}$

Now considered the "standard" formulation

- **Terminal cost**: $V_f(x) = x'Px$, from the **Riccati equation**:

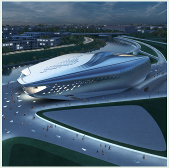
$$P = Q + A'PA - A'PB(B'PB + R)^{-1}B'PA$$

- **Terminal set**: $\mathbb{X}_f = \{x \in \mathbb{R}^n \mid V_f(x) \le \alpha\}$ with $\alpha > 0$ **suitably chosen** such that

$$x \in \mathbb{X} \qquad Kx \in \mathbb{U} \qquad \text{with } K = -(B'PB + R)^{-1}B'PA$$

Comments

- **Closed-loop** and **open-loop** trajectories **coincide**
- It is an **infinite-horizon optimal** formulation
- Often the **terminal constraint** is **not enforced**, but verified **a-posteriori** (**increasing** N if not satisfied)



Types of uncertainties

... The bare truth

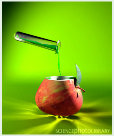
- The **true** controlled system **does not satisfy** $x^+ = Ax + Bu$
- The **true state** x is **not known exactly**

Additive uncertainty

- The **true system is modeled** as

$$x^+ = f(x, u) + w \qquad \text{with } f(x, u) = Ax + Bu$$

- The **disturbance** w is **unknown** but **bounded**, $w \in \mathbb{W}$ ($\mathbb{W}$ **compact and convex**)



Alternative LTV description (convex hull)

$$x(k+1) = A(k)x(k) + B(k)u(k) \qquad \text{with } \{A(k), B(k)\} = \sum_{i=1}^{M} \mu_i(k)\{A(i), B(i)\}$$

Closed-loop uncertain system under nominal MPC

Difference inclusion description

- The **true system** can be modeled as a **difference inclusion**

$$x^+ \in F(x,u) = \{f(x,u) + w \mid w \in \mathbb{W}\}$$

- If the **state** is **not precisely known**:

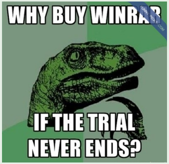
$$u = \kappa_N(x+e) \qquad \text{with } e \in \mathbb{E} \text{ (compact and convex)}$$

- The **closed-loop system** evolves as:

$$x^+ \in H(x) = \{f(x, \kappa_N(x+e)) + w \mid e \in \mathbb{E},\ w \in \mathbb{W}\}$$

with a generic solution denoted as $\phi_{ew}(k;x)$

Fundamental questions: if $\mathbb{W}$ and $\mathbb{E}$ *are small* sets



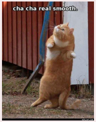
- Is $\mathbb{P}_N$ solvable at all times (**recursive feasibility**)?
- Does the following **robust stability condition** hold?

$$|\phi_{ew}(k;x)| \leq \beta(|x|, k) + \epsilon \qquad \text{with } \epsilon > 0$$

Inherent robustness of linear MPC

Properties of $\mathbb{P}_N$ for linear MPC [Grimm et al., 2004]

- The **optimal cost function** $V_N^0(\cdot)$ is continuous (in x)
- The **optimal MPC law** $\kappa_N(\cdot)$ is continuous (in x)



Robust asymptotic stability

- Grimm et al. [2004] showed that if:
  - there exists a **continuous Lyapunov function** for the nominal system $x^+ = f(x, \kappa_N(x))$, and
  - $\mathbb{P}_N$ is **feasible at all times**
- Then, for **any** $\epsilon > 0$ there **exists** $\delta > 0$ such that if $\{\mathbb{W}, \mathbb{E}\} \in \delta\mathbb{B}$:
  $|\phi_{ew}(k; x)| \leq \beta(|x|, k) + \epsilon$

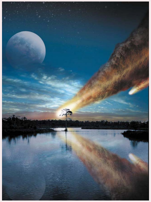
In [Grimm et al., 2004] recursive feasibility was **assumed**

... **Proved** in [Pannocchia et al., 2011a,b]

What is suboptimal MPC?

Why suboptimal MPC? ...A practical problem

- Despite its **convexity** (only for **linear MPC**), solving $\mathbb{P}_N(x)$ **up to optimality** may be difficult if a **short decision time** is allowed
- **Stability theory** assumed that $\mathbb{P}_N(x)$ is **solved exactly**
- What is the **impact** of using a **suboptimal solution** to $\mathbb{P}_N(x)$?



A neat suboptimal MPC framework [Scokaert et al., 1999]

- Given **current state** x, **previous control** sequence $\mathbf{u}^- = \{u^-(0), u^-(1), \ldots, u^-(N-1)\}$ and **state** sequence $\mathbf{x}^- = \{x^-(0), x^-(1), \ldots, x^-(N)\}$
- Build a **warm-start**: $\mathbf{u}_0 = \{u^-(1), \ldots, u^-(N-1), \kappa_f(x^-(N))\}$
- Perform **some iterations** to **improve** the **warm start**:

$$V_N(x, \mathbf{u}) \le V_N(x, \mathbf{u}_0)$$

Stability under suboptimal MPC

An additional ingredient

- To prove GAS, an **additional requirement** is enforced

$$V_N(x,\mathbf{u}) \leq V_f(x) \qquad \text{if } x \in r\mathbb{B} \subset \mathbb{X}_f$$

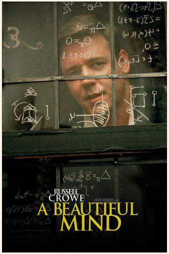
- $r > 0$ can be **arbitrarily small**: additional constraint **will not matter**

Sketch of stability proof.

- Consider the **extended state**: $z = (x, \mathbf{u})$
- The **successor suboptimal input sequence** $\mathbf{u}^+$ is a function of the x^+ and of the **warm-start**. Hence $\mathbf{u}^+ = g(x, \mathbf{u})$
- The **extended state** evolves as

$$\begin{bmatrix} x^+ \\ \mathbf{u}^+ \end{bmatrix} = \begin{bmatrix} Ax + B[I\ 0]\mathbf{u} \\ g(x,\mathbf{u}) \end{bmatrix} \qquad \text{or } z^+ = h(z)$$

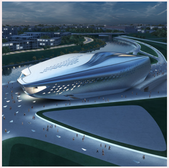
- $V_N(\cdot)$ is a **Lyapunov function** for $z^+ = h(z)$
- **Additional condition** implies GAS in the **non-extended state**



Inherent robustness of suboptimal MPC (1/2)

Comments on the suboptimal cost and control

- The **suboptimal control** is **not unique**, i.e. $\kappa_N(x)$ is **set-valued map**
- The **suboptimal cost** function $V_N(\cdot)$ is **not continuous** in x
- The proof of [Grimm et al., 2004] for **inherent robustness does not hold** for suboptimal MPC



New results [Pannocchia et al., 2011a,b]

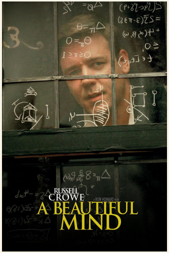
- **Suboptimal** MPC is **inherently robust**
- **Recursive feasibility** can be established
- **Optimal and suboptimal** MPC have the **same (qualitative)** stability properties

Inherent robustness of suboptimal MPC (2/2)

Sketch of robust stability proof.

- The **perturbed extended system** evolves as a **difference inclusion**

$$z^+ = H(z) := \{(x^+, \mathbf{u}^+) \mid x^+ = Ax + Bu(0; x+e) + w, \mathbf{u}^+ \in G(z)\}$$



- Show **exponential stability** in the **extended state**
- Prove that **exist** $\gamma \in (0,1)$ and $\mu > 0$

$$V_N(z^+) \leq \max\{\gamma V_N(z), \mu\}$$

- $V_N(\cdot)$ is continuous in z and implies **robust stability** in the **extended state**
- The **additional condition** implies robust stability in the **non-extended state**

Robust MPC design: an introduction (1/2)

An example [Rawlings and Mayne, 2009]

- (Nominal) system: $x^+ = x + u$, **without constraints** $\mathbb{X} = \mathbb{U} = \mathbb{R}$
- MPC **design**: $N = 3$, $\ell(x, u) = x^2 + u^2$, $V_f(x) = x^2$

Open-loop control vs feedback policies

OL Given **initial state** $x(0) = x$, **solve** for $\mathbf{u} = [u(0),\ u(1),\ u(2)]'$:

$$\mathbf{u}^0(x) = \begin{bmatrix} -0.615x & -0.231x & -0.077x \end{bmatrix}'$$

FB Use **dynamic programming** to obtain a **feedback policy**:

$$\boldsymbol{\mu}^0 = \begin{bmatrix} -0.615x(0) & -0.6x(1) & -0.5x(2) \end{bmatrix}'$$

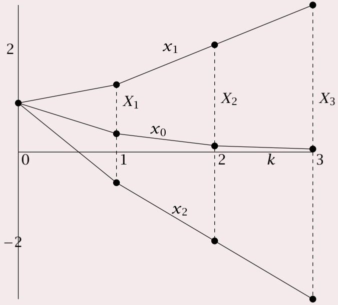
Evolution in the presence of uncertainties

- Same **nominal evolution** is obtained
- Considering **disturbances**: $x^+ = x + u + w$, **different trajectories** are obtained

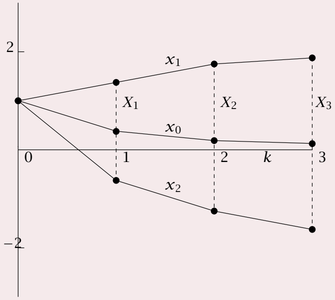
Robust MPC design: an introduction (2/2)

Trajectories in three cases

- Three **disturbance sequences**:
  - $\mathbf{w}^0 = \{0, 0, 0\}$
  - $\mathbf{w}^1 = \{1, 1, 1\}$
  - $\mathbf{w}^2 = \{-1, -1, -1\}$



(a) Open loop.



(b) Feedback.

- **Feedback policies** are clearly **more effective** against **disturbances**

Min-max MPC

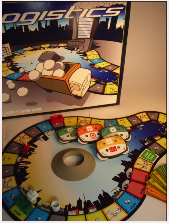
Conceptual framework

- **Prediction model** $x^+ = Ax + Bu + w$ with $w \in \mathbb{W}$ (compact)
- **Robustly invariant terminal** set $\mathbb{X}_f$ [Blanchini, 1999]
- **Open-loop min-max**: $\mathbf{u} = [\, u(0) \;\; u(1) \;\; \cdots \;\; u(N-1) \,]$

$$\min_{\mathbf{u}} \quad \max_{\mathbf{w}} V_N(x, \mathbf{u}, \mathbf{w}) \qquad \text{s.t.}$$
$$x(j+1) = Ax(j) + Bu(j) + w(j)$$
$$x(j) \in \mathbb{X}, \quad w(j) \in \mathbb{W}, \quad u(j) \in \mathbb{U} \text{ for } j = 0, \ldots, N-1$$
$$x(N) \in \mathbb{X}_f$$

- **Feedback min-max**: $\boldsymbol{\mu} = [\, \mu(x(0)) \;\; \mu(x(1)) \;\; \cdots \;\; \mu(x(N-1)) \,]$

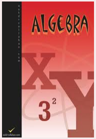
$$\min_{\boldsymbol{\mu}} \quad \max_{\mathbf{w}} V_N(x, \boldsymbol{\mu}, \mathbf{w}) \qquad \text{s.t.}$$
$$x(j+1) = Ax(j) + B\mu(x(j)) + w(j)$$
$$x(j) \in \mathbb{X}, \quad w(j) \in \mathbb{W}, \quad \mu(x(j)) \in \mathbb{U} \text{ for } j = 0, \ldots, N-1$$
$$x(N) \in \mathbb{X}_f$$



Tube-based MPC (1/3)

Set algebra

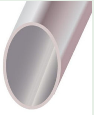
- Some notation
  - Set **addition**: $A \oplus B = \{a + b \mid a \in A,\ b \in B\}$
  - Set **subtraction**: $A \ominus B = \{x \in \mathbb{R}^n \mid \{x\} \oplus B \subseteq A\}$
  - Set **multiplication**: let $K \in \mathbb{R}^{m \times n}$. $KA = \{Ka \mid a \in A\}$



Outer-bounding tube

- **Uncertain linear system**: $x^+ = Ax + Bu + w,\ w \in \mathbb{W}$
- **Nominal system**: $z^+ = Az + Bv$
- **Affine feedback policy**: $u = v + K(x - z)$
- **Error**, $e = x - z$, evolves as: $e^+ = (A + BK)e + w = A_K e + w$
- If we set $z(0) = x(0)$, i.e. $e(0) = 0$, then

$$e(i) \in S_K(i) = \sum_{j=0}^{i-1} A_K^j \mathbb{W} \subseteq S$$

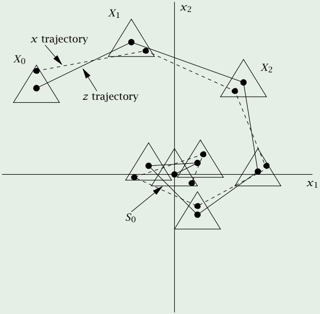


Tube-based MPC (2/3)

Constraint tightening

- **Constraints** on the **uncertain system**: $x(i) \in \mathbb{X}$, $u(i) \in \mathbb{U}$
- **Tightened** constraints on the **nominal system**:
  $z(i) \in \mathbb{Z} = \mathbb{X} \ominus S$, $v(i) \in \mathbb{V} = \mathbb{U} \ominus KS$

A sketch of nominal and uncertain trajectories



Tube-based MPC (3/3)

Nominal MPC problem with restricted constraints

$$\bar{\mathbb{P}}_N(z): \qquad \min_{\mathbf{v}} V_N(z, \mathbf{v}) \qquad \text{s.t.} \quad z^+ = Az + Bv$$
$$z(j) \in \mathbb{Z} \qquad \text{for all } j = 0, \dots, N-1$$
$$v(j) \in \mathbb{V} \qquad \text{for all } j = 0, \dots, N-1$$
$$z(N) \in \mathbb{Z}_f$$

Tube-based MPC

Initialization At time $k = 0$, set $z(0) = x(0)$

Step 1 Given current **augmented state** (x, z), solve $\bar{\mathbb{P}}_N(z)$ and obtain **nominal control** $v = \bar{\kappa}_N(z)$

Step 2 Apply **control**: $u = v + K(x - z)$

Step 3 Compute **nominal successor state**: $z^+ = Az + Bv$ and measure **successor state** x^+

Step 4 Replace $(x, z) \leftarrow (x^+, z^+)$, go to Step 1

Output feedback MPC: main definitions

True system and state estimator

- **Uncertain LTI** system

$$x^+ = Ax + Bu + w$$
$$y = Cx + v$$

- **Bounded disturbances**: $w \in \mathbb{W}$, $v \in \mathbb{V}$
- Simple **Luenberger observer**:

$$\hat{x}^+ = A\hat{x} + Bu + L(y - \hat{y}) \qquad \text{with } \hat{y} = C\hat{x}$$

- **Estimation error** $e := x - \hat{x}$ evolves as

$$e^+ = (A - LC)e + \tilde{w} \qquad \text{with } \tilde{w} := w - Lv \in \tilde{\mathbb{W}} := \mathbb{W} \oplus (-L\mathbb{V})$$

Output feedback MPC

- Solve $\mathbb{P}_N(\hat{x})$ and apply $\kappa_N(\hat{x})$

Nominal stability of output feedback MPC

Deterministic case

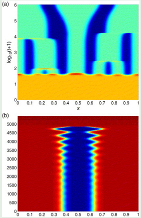
- In the **ideal situation**: $\mathbb{W} = \{0\}$ and $\mathbb{V} = \{0\}$: $e^+ = (A - LC)e$
- The origin of $e^+ = (A - LC)e$ is **exponentially stable**
- **Estimator state** evolves as: $\hat{x}^+ = A\hat{x} + Bu + LCe$

Main result [Scokaert et al., 1997]

- Let $\phi(k; x, e)$ be the **solution** at time k of $x^+ = Ax + B\kappa_N(\hat{x})$
- The following **asymptotic stability** condition holds:

$$|\phi(k; x, e)| \leq \beta(|(x, e)|, k) \qquad \text{for all } k \in \mathbb{N}$$

for any **initial state** $x \in \mathscr{C} \subset \mathscr{X}_N$ and **estimate error** $e \in \mathscr{E}$



Offset-free MPC based on disturbance model

Some reminders

- The **augmented** system ($x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p, d \in \mathbb{R}^{n_d}$)

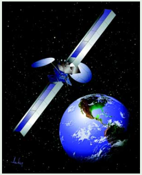
$$\begin{aligned} x^+ &= Ax + Bu + B_d d \\ d^+ &= d \\ y &= Cx + C_d d \end{aligned}$$

- **Observability** requirements

(A, C) observable $\qquad \text{rank} \begin{bmatrix} A-I & B_d \\ C & C_d \end{bmatrix} = n + n_d$

Tracked variables, target calculator and dynamic optimization

- **Controlled** variables: $r = Hy$, with $r \in \mathbb{R}^{p_r}$ and $p_r \leq \min\{p, m\}$
- Target calculator **chooses targets** (x_s, u_s) such that:
  $x_s = Ax_s + Bu_s + B_d d, \qquad \bar{r} = H(Cx_s + C_d d)$
- **Dynamic optimization** regulates **deviation variables**:
  $\tilde{x} = x - x_s \to 0, \qquad \tilde{u} = u - u_s \to 0$



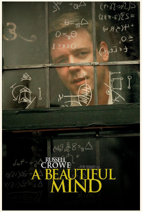
Zero offset [Muske and Badgwell, 2002, Pannocchia and Rawlings, 2003, Maeder et al., 2009]

Theorem statement

- Let $n_d = p$, assume that:
  - MPC **feasible** at all times, **unconstrained** for $k \geq \bar{k}$
  - **Closed loop** reaches **steady** values: (u_∞, y_∞), $(\hat{x}_\infty, \hat{d}_\infty)$, (x_s, u_s)
- Then, there is **zero offset** in r: $r_\infty = H y_\infty = \bar{r}$

Sketch of proof

- **Stability of the observer** implies that $L_d \in \mathbb{R}^{p \times p}$ is **full rank**:
  $\hat{d}_\infty = \hat{d}_\infty + L_d(y_\infty - C\hat{x}_\infty - C_d \hat{d}_\infty) \Rightarrow y_\infty = C\hat{x}_\infty + C_d \hat{d}_\infty$
- **Target** satisfies: $\bar{r} = H(Cx_s + C_d \hat{d}_\infty)$
- Since **constraints are inactive** (at steady state), $\tilde{u}_\infty = K\tilde{x}_\infty$
  Hence: $\tilde{x}_\infty = (A + BK)\tilde{x}_\infty \Rightarrow \tilde{x}_\infty = \hat{x}_\infty - x_s = 0 \Rightarrow \hat{x}_\infty = x_s$
- **Combining** all steps: $H(C\hat{x}_\infty + C_d \hat{d}_\infty) = H y_\infty = r_\infty = \bar{r}$



Equivalence of disturbance models and observer design

A debate: what is the *best choice* for (B_d, C_d)?

- There were **evidences** that $B_d = 0$, $C_d = I$ was a **bad choice** [Lundström et al., 1995, Muske and Badgwell, 2002, Pannocchia, 2003, Pannocchia and Rawlings, 2003, Maeder et al., 2009, Bageshwar and Borrelli, 2009]

A change of perspective

- Rajamani et al. [2004, 2009] argued that **two augmented systems** with **same** (A, B, C) and **different** (B_d, C_d) are two **non-minimal realizations** of the same system
- A **transformation matrix** T makes **them equivalent**
  $A_1 = \begin{bmatrix} A & B_{d1} \\ 0 & I \end{bmatrix}$, $B_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $C_1 = \begin{bmatrix} C & C_{d1} \end{bmatrix}$, $L_1 = \begin{bmatrix} L_{x1} \\ L_{d1} \end{bmatrix}$
  $A_2 = \begin{bmatrix} A & B_{d2} \\ 0 & I \end{bmatrix} = TA_1T^{-1}$, $B_2 = \begin{bmatrix} B \\ 0 \end{bmatrix} = TB_1$, $C_2 = \begin{bmatrix} C & C_{d1} \end{bmatrix} = C_1T^{-1}$, $L_2 = TL_1$
- Choose **any** (B_d, C_d) and **determine** (L_x, L_d) from data

$\mathscr{H}_\infty$ interpretation of disturbance models

$\mathscr{H}_\infty$ interpretation [Pannocchia and Bemporad, 2007]

- **System** $\mathscr{P}$ subject to a disturbance $w \in \mathbb{R}^{n+p}$

$$x^+ = Ax + Bu + B_w w \qquad B_w = [I_n \; 0]$$
$$y = Cx + \qquad + D_w w \qquad D_w = [0 \; I_p]$$

- **Design a dynamic observer** $\mathscr{L}$:

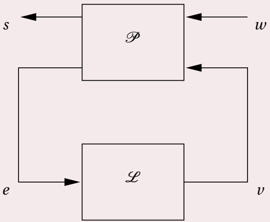
$$\xi^+ = A_L \xi + B_L e \qquad \text{with } e = y - \hat{y}$$
$$v = C_L \xi + D_L e$$

- Estimator in **closed loop**:

$$\hat{x}^+ = A\hat{x} + Bu + B_v v \qquad B_v = [I_n \; 0]$$
$$\hat{y} = Cx + \qquad + D_v v \qquad D_v = [0 \; I_p]$$

- **Equivalent** to the **augmented system** and **observer**:
  $A_L = I_p, \quad B_d = AB_v C_L, \quad C_d = D_v C_L$
  $L_x = AB_v D_L (I + D_v D_L)^{-1}, \quad L_d = B_L (I + D_v D_L)^{-1}$

- An $\mathscr{H}_\infty$ **observer** $\mathscr{L}$ such that the **DC-gain** $w \to s = e$ is zero is such that $A_L = I$



Alternative methods for offset-free MPC design (1/2)

Delta input form

- Assume for simplicity $r = y$. Define $\delta u(k) = u(k) - u(k-1)$, i.e. $u(k) = u(k-1) + \delta u(k)$, and the **augmented system**

$$\begin{bmatrix} x(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix} \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix} + \begin{bmatrix} B \\ I \end{bmatrix} \delta u(k)$$

$$y(k) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix}$$

- **Observer** to estimate $x_a(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix}$
- Solve the **dynamic optimization** penalizing $(y - \bar{r})$ and δu
- **Apply** $u(k) = u(k-1) + \delta u^0(k)$

Observations

- The system is **observable only** if $p \geq m$
- **Does not require** a target calculator
- **True input** $u(k-1)$ and its **estimate** $\hat{u}(k-1)$ may be **different**

Alternative methods for offset-free MPC design (2/2)

Velocity form

- $\delta x(k) = x(k) - x(k-1)$, $\delta u(k) = u(k) - u(k-1)$, $z = y - \bar{r}$
- Augmented system:

$$\begin{bmatrix} \delta x \\ z \end{bmatrix}^+ = \begin{bmatrix} A & 0 \\ CA & I_p \end{bmatrix} \begin{bmatrix} \delta x \\ z \end{bmatrix} + \begin{bmatrix} B \\ CB \end{bmatrix} \delta u$$

$$y - \bar{r} = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \delta x \\ z \end{bmatrix}$$

- **Observer** to estimate $x_a = \begin{bmatrix} \delta x \\ z \end{bmatrix}$
- Solve the **dynamic optimization** penalizing z and δu
- **Apply** $u(k) = u(k-1) + \delta u^0(k)$

Observations

- The system is **stabilizable only** if $p \leq m$
- **Does not require** a target calculator
- May show **windup issues** if the setpoint $\bar{r}$ is **not reachable**

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