Higher order method for non linear equations resolution: application to mobile robot control

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***Abstract*— In this paper a novel higher order method for the resolution of non linear equations is proposed. The particular application to the mobile robot navigation in an environment with obstacles is considered. The proposed method is based on the *embedded-relaxed* approach in which the dimension of the resolution space is augmented and a different and faster direction toward the root is computed. The method is proved to converge with higher order for the augmented resolution space of dimension 2 and 3. Finally, the method is applied to the problem of mobile robot navigation between obstacles considered as repulsive potentials.**

I. Introduction

The problem of finding numerical solutions of non linear equations is typical in control problems and arises in several different applications. A novel higher order method, named *High Order Derivative method* (*HOD*), for the resolution of non linear equations is herein proposed. The proposed method is based on the *embedded-relaxed* approach proposed in [1] in which the dimension of the resolution space is augmented and a different and faster direction toward the root is computed. With higher order methods we refer to methods with convergence rate larger than two [2]. The proposed method will be proved to converge with higher order for an augmented resolution space of dimension 2 (order 3) and 3 (order 4). While a general proof for the dimension n is still under study.

Between the large number of possible application in which the proposed approach can be used, the mobile robot navigation problem has been considered. The motion planning problem based on potential fields is a well known approach widely used since decades [3], [4]. An extensive explanation on the potential fields approaches for motion planning can be found in [5] and [6]. The description of the potential field approach for robot navigation is out of the scope of this paper in which the particular problem has been considered as a simple and intuitive case-study for the proposed HOD method.

For the particular application several different approaches based on the same HOD method are described. In particular a strategy to avoid local non desired minimum in a potential fields is proposed.

The paper is organized as follows. In section II the Higher Order Derivative method is described. In section III the order of convergence is proved for dimensions $n = 2$ and $n = 3$. Comparisons with the Newton methods are also reported in this section.

The particular case-study application considedered in this paper is described in section IV. Finally in section V simulations results have been reported.

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II. Higher order Derivative method

The proposed method is inspired by the work of Germani et al. [1] in which an *embedded-relaxed* approach is described. In particular, the problem of finding the solution of the non linear equation $f(x) = 0$ is solved by embedding the equation in the n-equations system

$$\begin{cases} f(x) = 0 \\ f(x)^2 = 0 \\ \vdots \\ f(x)^n = 0, \end{cases} \tag{1}$$

whose solutions are the same of the initial scalar equation.

In this paper a similar approach is applied embedding the equation $f(x) = 0$ (with $f \in \mathbb{C}^n$) in the following n-dimensional system

$$\begin{cases} f(x) = 0 \\ f(x)f'(x) = 0 \\ f(x)\left(f'(x)\right)^2 = 0 \\ \vdots \\ f(x)\left(f'(x)\right)^{n-1} = 0, \end{cases} \tag{2}$$

where $f'(x) = \frac{df}{dx}(x)$.

The proposed embedding approach is based on the observation that if we are interested in finding roots of $f(x) = 0$ we may focus on the minima of $f^2(x)$ whose derivative is given by $f(x)f'(x)$.

Given system (2) a n-degree Taylor expansion of functions $f(x)\left(f'(x)\right)^i,\ i = 0, \dots, n-1$ around $\tilde{x}$ is considered.

Let $\tilde{\mathbf{f}} = \begin{bmatrix} f(\tilde{x}) & f(\tilde{x})f'(\tilde{x}) & \dots & f(\tilde{x})\left(f'(\tilde{x})\right)^{n-1} \end{bmatrix}^T$, $\mathbf{f} = \begin{bmatrix} f(x) & f(x)f'(x) & \dots & f(x)\left(f'(x)\right)^{n-1} \end{bmatrix}^T$. Finally, let $\mathbf{v} = \begin{bmatrix} (\tilde{x} - x) & \dots & (\tilde{x} - x)^n \end{bmatrix}^T$.

System (2) is approximated with

$$\tilde{\mathbf{f}} = \mathbf{f} + A\mathbf{v}, \tag{3}$$

where

$$A = \begin{bmatrix} f^{(1)}(x) & \dots & \frac{1}{n!}f^{(n)}(x) \\ \vdots & \dots & \vdots \\ \left(f(f')^{n-1}\right)^{(1)}(x) & \dots & \frac{1}{n!}\left(f(f')^{n-1}\right)^{(n)}(x) \end{bmatrix}, \tag{4}$$

where $h^{(i)}(x) = \frac{d^i}{dx^i} h\Big|_x$ is the i-th derivative of function h computed in x. The iterative method is obtained by considering $x = x_k$ and $\tilde{x} = x_{k+1}$ and imposing $f(\tilde{x}) = 0$. The relaxation phase is then considered by relaxing constraints between components of $\mathbf{v}$ (powers of $(x_{k+1} - x_k)$) and considering the system

$$A \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = -\mathbf{f}, \tag{5}$$

By solving the system (5) the first component of y is obtained and the iteration step is given by

$$x_{k+1} = x_k + y_1(k), \tag{6}$$

where $y_1(k)$ is the first component of the solution of (5) at k-th iteration.

III. Order of convergence

Given the iterative method $x_{k+1} = g(x_k)$ for the solution of problem $g(x) = x$ we have that

Def. *1:* the sequence $\{x_i\}$ is said to *converge to α with order $p \geq 1$* if

$$\lim_{i\to\infty} \frac{|x_{i+1} - \alpha|}{|x_i - \alpha|^p} = \gamma > 0$$

Furthermore, it is well known that

Theorem *1:* Let $\alpha \in [a, b]$ be the solution of $g(x) = x$, where $g \in C^p[a, b]$ with $p \geq 2 \in \mathbf{N}$. If

$$g^{(1)}(\alpha) = g^{(2)}(\alpha) = \cdots = g^{(p-1)}(\alpha) = 0, g^{(p)}(\alpha) \neq 0$$

there exist $\rho > 0$ such that for any $x_0 \in [\alpha - \rho, \alpha + \rho]$ the sequence $\{x_i\}$ generated by $x_{k+1} = g(x_k)$ converges to α with order p.

In our case, the iteration is given by (6), hence $g(x) = x + y_1$. For $n = 2$ we have that

$$A = \begin{bmatrix} f_1 & f_2 \\ f_1^2 + f_0 f_2 & 3f_1 f_2 + f_0 f_3 \end{bmatrix},$$

where for simplicity of notation $f_i = f^{(i)}(x)$.

Notice that method proposed in [1] is not applicable when $f'(x) = 0$ while in our case if $f'(x) = 0$ we have

$$A = \begin{bmatrix} 0 & f_2 \\ f_0 f_2 & f_0 f_3 \end{bmatrix},$$

with determinant equal to $-f_0 f_2^2$. Hence, when $f_1 = 0$ and $f_2 \neq 0$ our method is still applicable since matrix A is invertible. The main result of this paper os the following theorem.

Theorem *2:* The Higher Order Derivative method with $n = 2$ has at least order of convergence 3 and for $n = 3$ has order at least 4.

Proof: For space limitation and technicality of the proof not all explictic calculations have been reported.

By solving system (5) for $n = 2$, the iteration function $g(x)$ is given by

$$g(x) = x - \frac{f_0 \left(2 f_1 f_2 + f_0 f_3\right)}{2 f_1^2 f_2 + f_1 f_0 f_3 - f_0 f_2^2}.$$

Let α be a solution of $f(x) = f_0(x) = 0$, notice that $g(\alpha) = \alpha$. Let $N = 2f_1 f_2 + f_0 f_3$ and $D = 2f_1^2 f_2 + f_0 f_1 f_3 - f_0 f_2^2$, $g(x)$ can be written as

$$g(x) = x - f_0 \frac{N}{D}$$

. Notice that $N(\alpha) = 2f_1 f_2$, $D(\alpha) = 2f_1^2 f_2$, $N'(\alpha) = 2f_2^2 + 3f_1 f_3$ and $D'(\alpha) = 3f_1 f_2^2 + 3f_1^2 f_3$. Hence we have

$$g'(\alpha) = 1 - f_1 \frac{N(\alpha)}{D(\alpha)} = 1 - f_1 \frac{2f_1 f_2}{2f_1^2 f_2} = 0.$$

Furthermore,

$$\begin{aligned} g^{(2)}(\alpha) &= -2f_1 \left(\frac{N}{D}\right)'\Bigg|_\alpha - f_2 \frac{N(\alpha)}{D(\alpha)} \\ &= -2f_1 \frac{N'(\alpha) D(\alpha) - N(\alpha) D'(\alpha)}{D'^2(\alpha)} \\ &= -2f_1 \frac{-2f_1^2 f_2^3}{4f_1^4 f_2^2} - f_2 \frac{2f_1 f_2}{2f_1^2 f_2} = 0. \end{aligned}$$

It can also be computed that $g^{(3)}(\alpha) = \frac{f_1 f_3 + 3f_2^2}{2f_1^2}$. Hence the proposed method for $n = 2$ has at least order 3.

For $n = 3$ we obtain $g(\alpha) = \alpha$, $g'(\alpha) = g^{(2)}(\alpha) = g^{(3)}(\alpha) = 0$ while

$$g^{(4)}(\alpha) = \frac{3f_2^4 + 2f_1^2 f_3^2 + 2f_1 f_2^2 f_3 - f_1^2 f_2 f_4}{f_1^3 f_2}.$$

Hence the proposed method for $n = 3$ has at least order 4. ■

Remark 1: Obviously, more complicated results are obtained for larger dimensions n. A general proof that the order of convergence of the method applied to a n dimensional system is $n + 1$ is currently under study.

A. Comparisons with Newton method

Consider the case of a polynomial $p(x) = x^4 + 4x^3 - 3x^2 + 8x - 5$. The Newton method and the HOD method have been applied to $p(x)$ with initial condition $x_0 = 1.5$. The iterations evolution $(x_{k+1} = g(x_k))$ of the classical Newton method and the HOD method are reported in figure 1. The blue line is the function $p(x)$. The green dotted line represents the Newton method evolution. The Newton method is also known as the tangent method, indeed the dotted line is tangent to the curve $p(x)$ in x_k. The continuous red line represents the evolution of the Derivative method. Notice that, for the given $p(x)$ the Derivative method computes, in one step, an approximation solution close to the one obtained with the Newton method in two steps (Newton method is known to have order of convergence 2).

As mentioned, the proposed method is based on the embedding-relaxing approach in which a higher dimensional system is considered in a higher dimensional space. A

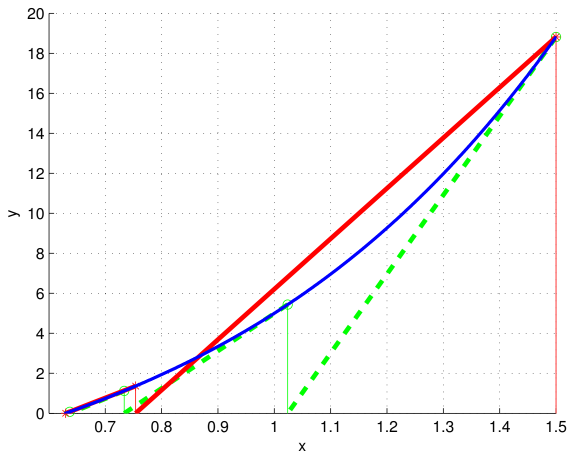

Fig. 1. Comparison of the evolution of the Newton iteration method and the higher order Derivative method. The blue line is the function $p(x)$, the green dotted line represents the Newton method evolution and the continuous red line represents the evolution of the Derivative method.

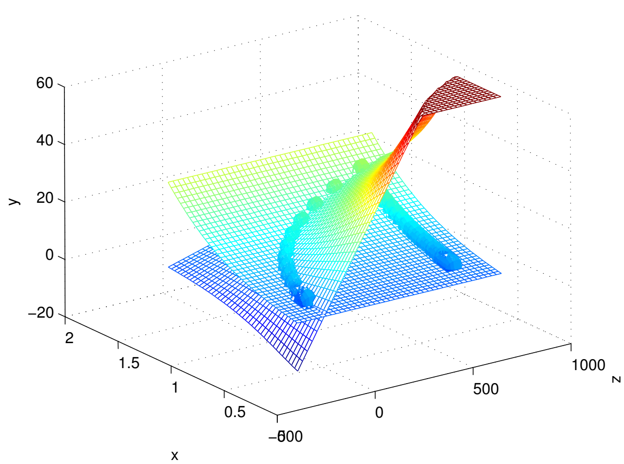

Fig. 2. Simulation of the evolution of the two methods in the augmented space. The Newton method evolution is represented on a plane in the higher dimensional space while the Derivative method evolves in the whole space.

simulation of the (interpolated) evolution of the two methods in the 2-dimensional augmented space is reported in figure 2. The Newton method evolution is represented on a plane in the higher dimensional space while the Derivative method evolves in the whole space. Notice that the Derivative method evolves along the intersection of the surfaces $f(x)\,(f'(x))^i$, with $i = 0,\,1,\,\ldots,\,n$.

In particular, steps of the methods in the augmented space (without interpolation on the intermediate points) are reported in figure 3.

IV. Application to mobile robot control in constrained environment

In this section, the particular application to mobile robot navigation in constrained environment is considered. The HOD method is applied to allow the robot to navigate

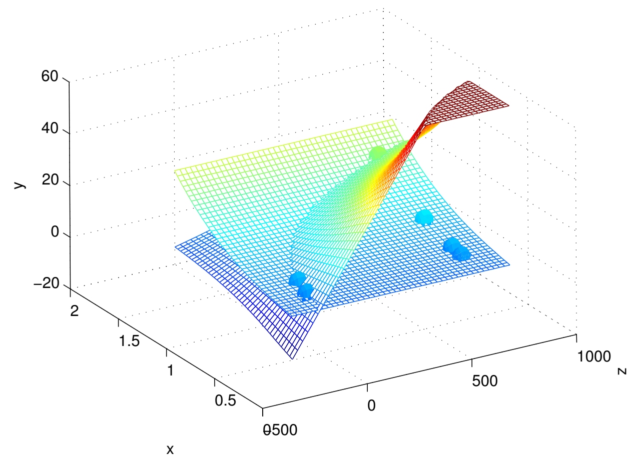

Fig. 3. Single steps of the Derivative and the Newton methods in the augmented space.

avoiding obstacles or forbidden sectors. The environment is a 2-dimensional that we consider as the complex plane in the following represented by real components x and y or $z = x + iy$.

Let z_i be the positions of the obstacles centers or points that the robots may want to avoid for security or safety reasons. Let us consider the complex function $F(z) = \sum_{i=1}^{n} \frac{1}{z - z_i} = \frac{p'(z)}{p(z)}$ where $p(z) = \prod_{i=1}^{n} z - z_i$. Hence, $F(z) = \frac{d}{dz} \log p(z)$. Considering F as a navigation function we would like the robot to navigate far from the points z_i, i.e. we would like the robot to move where $\left|\frac{p'(z)}{p(z)}\right| \le c$ for a given value $c > 0$.

In particular we may want to drive the robot along the curves

$$p'(z)p'(-z) = c^2 p(z)p(-z), \qquad (7)$$

hence we want to solve a problem of the form $f(z) = p'(z)p'(-z) - c^2 p(z)p(-z) = 0$ where $z \in \mathbb{C}$.

With the described method we allow the robot to move towards a solution of (7) with higher convergence order. By changing the value of c the robot is able to navigate between the iso-potential curves as obtained with simulations whose results are reported and described in next section.

V. Simulations results

In this section the simulations results of the Derivative method and its application to the robot motion planning problem are reported.

A. *Derivative method simulations results*

The derivative method has been compared to the Newton method in section III-A. In this section we report and comment some simulation results obtained on the HOD method.

Given $f(z) = z^3 - \sin(z)$, starting from $z_0 = 4$, requiring a precision of 10^{-12} we have simulated the Derivative method for different values n. Simulations results are reported in the following table in which n represents the dimension

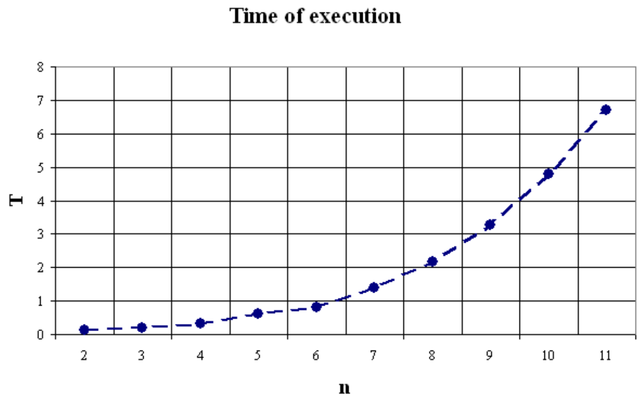

Fig. 4. Relation between the time of execution and the dimension of the system to be solved for $f(z) = z^3 - \sin(z)$, starting from $z_0 = 4$, requiring a precision of 10^{-12}.

of the augmented space (we conjecture that the method has order $n+1$), T the time of execution expressed in seconds, N_s the number of steps required to approximate the solution and $f(\bar{z})$ is the value of the function f computed on the approximated solution obtained by the algorithm. The solution found by the algorithm is in 0.9286.

n	N_s	T	$f(\bar{z})$
2	6	0.12	0
3	5	0.18	$-1.11\,10^{-16}$
4	4	0.31	0
5	4	0.61	0
6	3	0.88	$-1.11\,10^{-15}$
7	3	1.3	$-6.66\,10^{-16}$
8	3	2.17	$8.88\,10^{-16}$
9	3	3.26	$-6.66\,10^{-16}$
10	3	4.79	$-3.33\,10^{-16}$
11	3	6.69	0

In figure 4 the relation between the time of execution and the dimension of the system to be solved are reported in a graphic. As expected, the computational time increases with the dimension of the system to be solved (represented by n). Furthermore, in this case the method applied with $n \geq 6$ is able to find an approximation of the solution with the required precision with 3 steps. Hence, by fixing the number of steps N_s the best choice is obviously to choose the lowest order method (in our case $n = 6$) in order to minimize the computational time.

B. Simulation of the application to Robot motion planning

In the first group of simulations, we consider a simplyfied version of of equation (7): we add the factor $z-a$ on the left hand-side and we remove $p'(z)$ in order to avoid the presence of multiple local minima. Hence, the new simplified equation is In particular we may want to drive the robot along the curves

$$(z-a) = c^2 p(z)p(-z), \tag{8}$$

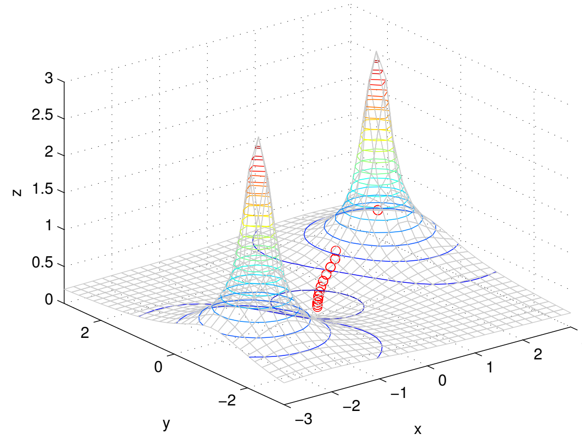

Fig. 5. Evolution of the robot in the constrained environment

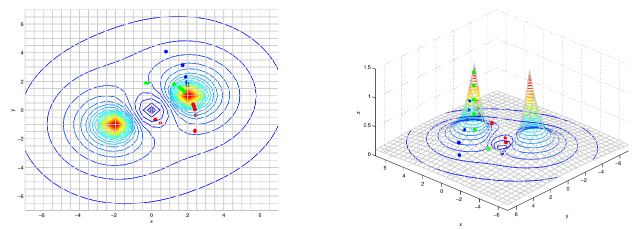

Fig. 6. Evolution of the robot in the constrained environment for different initial configurations

The local minima escape problem will be described at the end of the section.

In the first group of simulation the polynomial $p = z^4 - 4iz^2 + 8z^2 + 12 + 16i$ and $a = 0$ have been considered. The value c of level curves has been decreased of a factor 10 starting from the value at the initial configuration, e.g. $z_0 = 1-2i$. The evolution of the robot is then forced between level curves to the origin through the Derivative method that is re-applied anytime the value c is changed. A representation of the evolution of the robot in a constrained environment reported in figure 5.

In another simple environment, different initial configurations have been considered and the corresponding trajectories are represented with different colors in figures 6.

In figures 7 a more complicated scenario with more obstacles is represented with two different initial configurations.

Another possible application of the method is to consider a zero that may vary in time. Indeed, by considering a fixed initial configuration z_0 and let the zero varying along a curve, or a straight line we obtain evolutions as the ones reported in figures 8. In this case we have considered a fixed initial configuration $z_0 = -2-2i$ and a zero varying along a straight line ($z = -2,\ -1,\ 0,\ 1,\ 2$). Every evolution has been plotted with different colors and it can be noticed that the

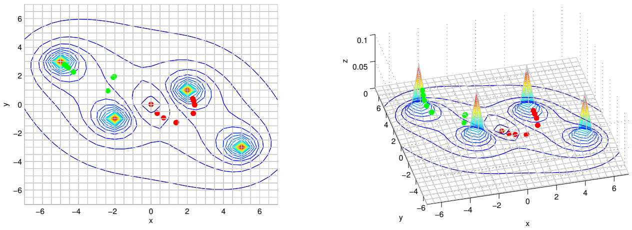

Fig. 7. Evolution of the robot in the constrained environment for different initial configurations

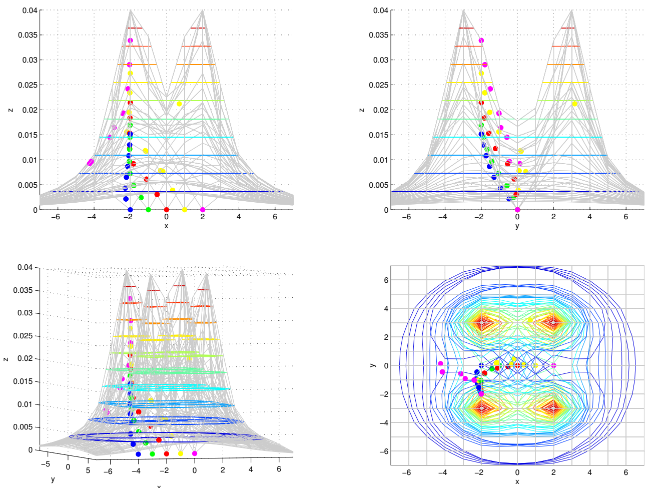

Fig. 8. Evolution of the robot in the constrained environment

five evolutions reach the five zeros as required. In figures 8, different points of view of the same scenario have been represented.

Two different techniques have been described above: the first is based on the variation of the iso-potential curves value c that decreases to zero, the second is based on the variation of the target configuration (zero of the function). This two techniques may be combined to allow more complicated trajectories between obstacles. In figure 9 a simulation of the combined approach applied to the same scenario considered above is reported: c is decreased of a factor five while the zero varies along a straight line (zero= -2, -1, 0, 1, 2).

Finally, another example of the combined application for the robot exploration around obstacles is reported in figure 10. In this case the initial configuration is $z_0 = -3.5 - i$, the value c is scaled of a factor 40 while the zero varies along the cirumference centered i nthe origin with radius 6 starting from $z = -6i$. With this particular approach the robot is able to navigate between obstacles trying to reach a point that moves along a circumference containing the obstacles.

C. Escape from local minima

As mentioned at the beginning of previous subsection, equation (7) have several local minima. In case of $c = 0$ the minima are the roots of $p'(z)$. Hence, the HOD method

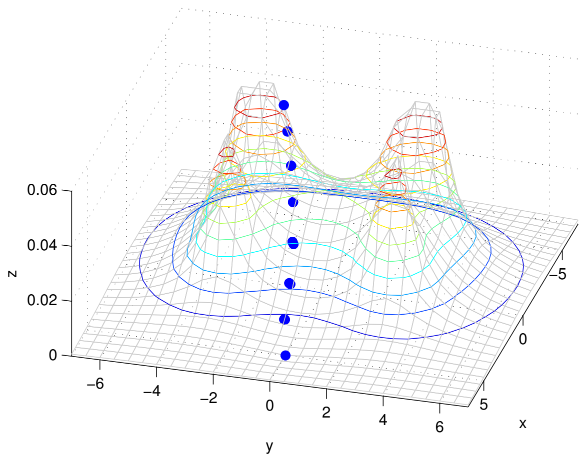

Fig. 9. Evolution of the robot in the constrained environment with a combined approach.

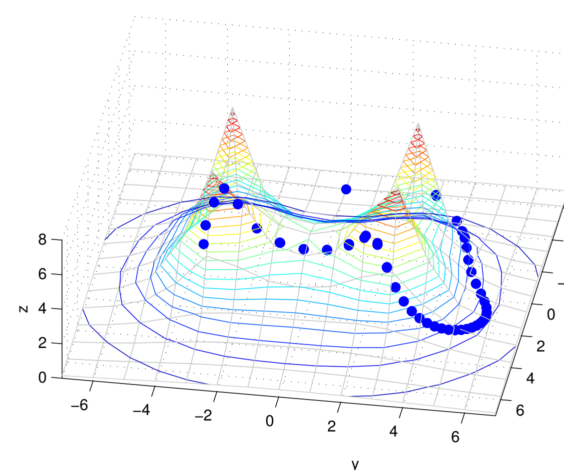

Fig. 10. Exploration of the robot around obstacles with the combined approach.

may drive the robot in one of the local minimum that may be different to the zero $z = a$. In order to avoid this problem, any time the algorithm find a local minimum different from the desired one the equation (7) is modified accordingly. In particular, another peak is added closed to the local minimum to create a different fictitious environment in which the current configuration is not a minimum anymore while the desired configuration is still a zero of the equation.

For example, by considering $z_1 = 3 + 2i$ and $z_2 = 4 - 3i$ we have

$$f(z) = (-4z^2+48-14i)z+c*(z^4+(-12+12i)z^2+323-36i),$$

starting from $z_0 = 3 + i$ and let c decreasing of a factor 10, the HOD method drive the robot to $3.5 - 0.5i$ that is a zero of $p'(z) = -4z^2 + 48 - 14i$ as reported in figure 11 and 12.

Choosing the virtual environment as described above the new evolution is described in figures 13 and 14 and the origin

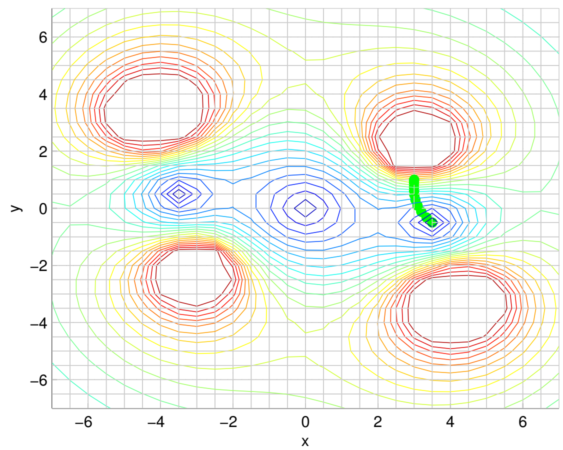

Fig. 11. The robot may reach a local minimum different from the desired one.

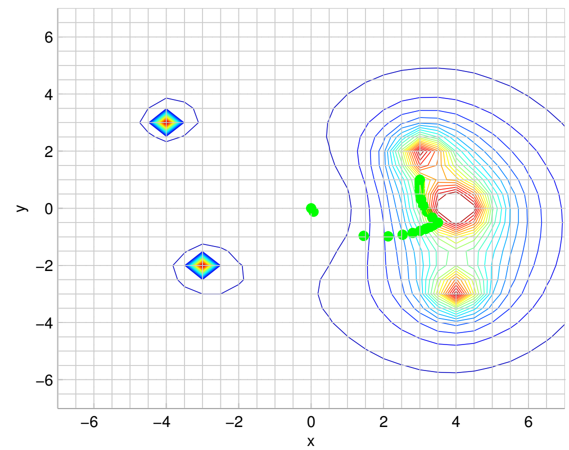

Fig. 13. The robot escapes a local minimum and reaches the origin in the fictitious environment.

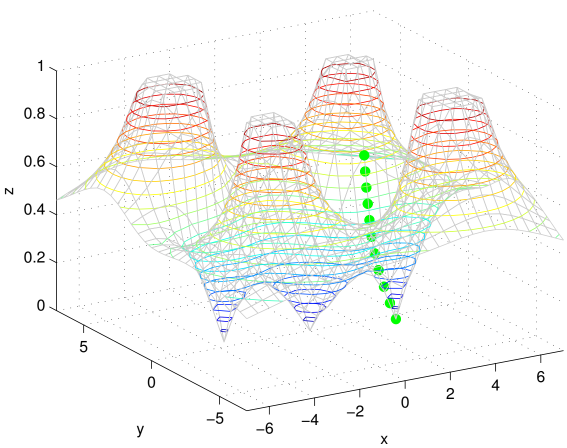

Fig. 12. The robot may reach a local minimum different from the desired one (lateral view).

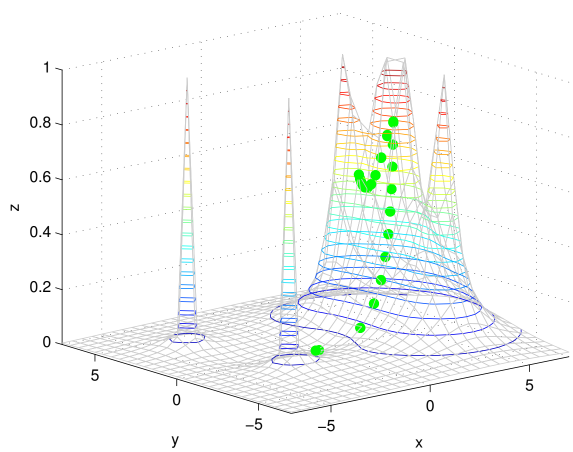

Fig. 14. The robot escapes a local minimum and reaches the origin in the fictitious environment (lateral view).

has been reached.

VI. Conclusions and Future Works

In this paper a novel higher order method for the resolution of non linear equations has been proposed. An *embedded-relaxed* approach in which the dimension of the resolution space is augmented and a different and faster descent direction to the root is computed has been described. Proof of convergence and order rate for an augmented resolution space of dimension 2 and 3 have been reported. A proof of the general rate of convergence is still under study.

The method has been applied to the problem of mobile robot navigation between obstacles considered as repulsive potentials. A strategy to avoid local minima has also been proposed. Finally, several simulation results have been reported.

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