On the Reachability of Quantized Control Systems

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Abstract— **In this paper we study control systems whose input sets are quantized, i.e. finite or regularly distributed on a mesh. We specifically focus on problems relating to the structure of the reachable set of such systems, which may turn out to be either dense or discrete. We report results on the reachable set of linear quantized systems, and on a particular but interesting class of nonlinear systems, i.e. nonholonomic chained–form systems. For such systems, we provide a complete characterization of the reachable set, and, in the case the set is discrete, a computable method to completely and succinctly describe its structure. Implications and open problems in the analysis and synthesis of quantized control systems are addressed.**

I. Introduction

In this paper we consider discrete-time systems of the type

$$x^+ = g(u,x),\ x \in \mathbb{R}^n,\ u \in \mathcal{U} \subset \mathbb{R}^m \tag{1}$$

where the input set, U, is quantized, i.e. finite or with values on regular meshes in $\mathbb{R}^m$. Quantized control systems (QCS) arise in a number of applications because of many physical phenomena or technological constraints. In the control literature, quantization of inputs has been mostly regarded as an approximation-induced disturbance to be rejected ([1], [2]). Typical results in this spirit are those provided by [3], who show how a nonlinear system with quantized feedback, whose linear approximation (without quantization) has an asymptotically stable solution, has uniformly ultimately bounded solutions; and how such bounds can be made small at will by refining quantization sufficiently.

A different viewpoint, that has been championed by D. Delchamps in the early 90's ([4], [5]), is that quantization is a deterministic, memoryless nonlinear phenomenon that may affect inherent properties of the system in very specific ways, and that its study should, and indeed can in some cases, be performed directly. This approach is particularly meaningful when quantization is rough, or when it is introduced on purpose in order to reduce the technological complexity of the control systems. The latter concern is very relevant in many present-day control systems, such as e.g. in mass-produced embedded systems (where electronics cost reduction is at a premium) or in distributed control systems. Recently, some attention has been focused on QCS as specific models of hierarchically organized systems with interaction between continuous dynamics and logic ([6], [7]). As a consequence of taking such viewpoint, the focal point of research is to understand how to design a quantized system, rather than assessing robustness of a continuous design with respect to quantization.

While [4] focused on observability with quantized outputs, [5],[8], [6] and [7] addressed the stabilization problem. Authors of the latter paper provide a result on the optimal (coarsest) quantization for asymptotically stabilizing a linear discrete–time system, that turns out to require a countable symmetric set of logarithmically decreasing inputs, namely $\mathcal{U} = \{\pm u_i : u_{i+1} = \rho u_i, -\infty \le i \le +\infty\} \cup \{0\}$. Although this choice (and the corresponding partition induced in the state space) captures the intuitive notion that coarser control is necessary when far from the goal, it still needs input values that are arbitrarily close to each other near the equilibrium.

An observation common to many papers on stabilization with quantized control is that, if the available quantized control set is finite, or countable but nowhere dense (in the natural topology of $\mathbb{R}^m$) then stability can only be achieved in a weak sense — be it ultimate boundedness ([3]), containability ([6]), or practical stability ([7]).

The focus of our paper is on the study of particular phenomena that may appear in QCS, which have no counterpart in classical systems theory, and that deeply influence the qualitative properties and performance of the control system. These concern the structure of the set of points that are reachable by system (1), and particularly its density. While some understanding of the structure of the reachable set for quantized linear systems has been reached recently ([9]), the general nonlinear case remains largely unexplored, and probably quite hard to attack. In this paper, we consider a particular but important class of nonlinear systems, i.e. chained-form systems. This class has been introduced by [10] as a canonical form for continuous-time driftless nonholonomic systems, and has since been used extensively in the automatic control literature for modeling and controlling systems that range from wheeled vehicles (with an arbitrary number of trailers) to satellites (see e.g. [11], [12], [13], [14], [15], [16], [17]). However, to our knowledge, properties of this class of systems in under quantized inputs have not yet been considered in the literature.

The main contribution of this paper is probably theorem 9, which describes the structure of the reachable set for chained form systems, under quantized control inputs. Specifically, this theorem provide conditions for the reachable set to be discrete, or otherwise to be dense in the state space, or to have a compound structure. When the discrete case applies, we show that the reachable set possesses a lattice structure, for which we provide a complete description

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by a finitely computable algorithm: this is instrumental to devising steering methods for the system based on integer programming techniques. Density of the reachable space is shown to obtain only under some irrationality conditions on the control values: the practical implications of this result, as a limit case for increasing quantization resolution, are also discussed.

In the paper, we first provide few examples that illustrate differences with classical control systems, and some definitions that extend classical notions of reachability to systems with quantized input sets (section II). In section III we study linear QCS. In particular, in III-A we report on recent results of [9] that apply to the dense synthesis problem, while in III-B we provide some new analysis results for simple linear systems, which are basic for later developments. In section IV we provide a complete solution for chained–form systems.

II. First definitions and examples

We consider systems defined as follows

Definition 1: A system is a quintuple $(\mathcal{X}, \mathcal{T}, \mathcal{U}, \Omega, \mathcal{A})$, *where* $\mathcal{X}$ *denotes the configuration set,* $\mathcal{T}$ *an ordered time set,* $\mathcal{U}$ *a set of admissible input symbols (possibly depending on the configuration),* Ω *a set of admissible input words formed by symbols in* $\mathcal{U}$ *and* $\mathcal{A}$ *is a state–transition map* $\mathcal{A} : \mathcal{T} \times \Omega \times \mathcal{X} \to \mathcal{X}$. *Denote* $\mathcal{A}_{t,\omega}(x) = \mathcal{A}(t, \omega, x)$, *with composition by concatenation* $\mathcal{A}_{t_1,\omega_2} \circ \mathcal{A}_{t_0,\omega_1}(x_0) = \mathcal{A}(t_1, \omega_2, \mathcal{A}(t_0, \omega_1, x_0))$.

In particular, we will focus here on $\mathcal{T} = \mathbb{N}$, not only because we are interested in digital control applications, but also because most interesting effects of quantization on reachability properties of systems appear to be linked to discrete time. Indeed, for instance, Raisch [18] has shown by optimal control arguments that the reachable space of continuous time LTI systems under quantized control coincides with that of the same system under continuous control (provided that controls have the same bounds componentwise). Similar results may be expected to hold for more general systems, as - roughly speaking - in continuous time one can choose to switch between different levels of quantized control at any time - basically allowing a pulse-width modulation (PWM) of signals.

A system as in definition 1 with both $\mathcal{X}$ and $\mathcal{U}$ discrete sets essentially represents a sequential machine or an automaton, while for $\mathcal{X}$ and $\mathcal{U}$ continuous sets, a discrete–time, nonlinear control system is obtained. We are interested in studying reachability problems that arise when $\mathcal{X}$ has the cardinality of a continuum, but $\mathcal{U}$ is discrete, i.e. when inputs are *quantized*. The following example motivates the generality of the definition above with a specific robotics application.

Example 1. We will consider the discrete analogue of a well known continuous nonholonomic system, which is the plate–ball system (see e.g. [19], [20], [21]). A ball rolls without slipping between two parallel plates, of which one is fixed and the other one translates. If the moving plate is driven along a closed trajectory, in particular e.g. it is translated to the right by some amount, then forward, left, and backward by the same amount, the same will happen to the ball center, which will end up in the same initial position. However, the final orientation of the sphere will be changed by a net amount. Indeed, it can be shown ([22]) that an arbitrary orientation in $SO(3)$ can be reached by rolling arbitrary pairs of non–isomorphic surfaces, which fact was used as a basis for building simplified dextrous robot hands.

Consider now a similar experiment with a polyhedron replacing the ball. For practical reasons, possible actions on this system are restricted to be rotations about one of the edges of the face lying on the plate, by exactly the amount that brings an adjacent face on the plate ([23], [24]). A generic configuration of the polyhedron can be described by giving the index of the face sitting on the plate, the position of the projection on the plate of the centroid, and the orientation of the projection of an inner diagonal of the cube. Hence, the configuration set is represented by the manifold $\mathcal{X} = \mathbb{R}^2 \times S^1 \times \mathcal{F}$, where $\mathcal{F}$ denotes the set of faces of the polyhedron. Given the discrete nature of input actions, we take $\mathcal{T} = \mathbb{N}_+$. For a given face $F \in \mathcal{F}$, and for all states with F on the plate ($x = (v, \theta, F)$, $v \in \mathbb{R}^2$, $\theta \in S^1$), the set of admissible symbols is the subset of faces adjacent to F, and Ω is the set of all sequences of adjacent faces starting with a face adjacent to F. Finally, $\mathcal{A}_\omega(x)$ is the configuration reached at the end of a sequence $\omega \in \Omega$ admissible at x. $\triangleleft$

Definition 2: A configuration x_f *is reachable from* x_0 *if there exists a time* $t \in \mathcal{T}$ *and an admissible input string* $\omega \in \Omega$ *that steers the system from* x_0 *to* $x_f = \mathcal{A}_{t,\omega}(x_0)$.

In the following we shall denote by R_x the reachable set from x, i.e. the set of configurations that can be reached from x. For differentiable systems, the notion of *reachability from* x is conventionally understood as $R_x = \mathcal{X}$. For discrete–time systems with quantized inputs, however, Ω is a subset of all possible finite sequences ω of symbols in the discrete set $\mathcal{U}$, hence R_x is a countable set and, in the general case that the configuration set $\mathcal{X}$ has the cardinality of a continuum, it will not make sense checking whether R_x equals $\mathcal{X}$.

Example 1–b. The set of configurations that can be reached starting from a given configuration of the polyhedron of Example 1, in a large but finite number of steps N, may have different characteristics. Consider for instance (intuitively, or by simulation) positions reached by the centroid of different polyhedra after N steps: only points lying on a regular grid can be reached by rolling a cube, while for a generic parallelepiped or pyramid they tend to fill the plane as N grows. Also, orientations obtained by rolling the cube or the parallelepiped are only multiples of $\pi/2$, while orientations reached by the generic pyramid tend to fill the unit circle as N grows (see [23], [24]). $\triangleleft$

Notice that the possibility that the reachable set of a quantized control system is discrete, separates such systems from differentiable systems; on the other hand, the possibility of having a dense reachable set distinguishes quantized control systems from classical finite–state machines. The structure of reachable sets will be described in the further assumption that $\mathcal{X}$ is a metric space with distance $d(x_1, x_2)$. We introduce a concept of *approachability* as

Definition 3: A configuration x_f can be approached *from x_0 if $\forall \epsilon$, $\exists t \in \mathcal{T}$, $\exists \omega \in \Omega$ such that $d(\mathcal{A}_{t,\omega}(x_0), x_f) < \epsilon$. We say that the system is* locally approachable from x_0 *if the closure of the reachable set R_{x_0} contains a neighborhood of x_0, and is* approachable from x_0 *if the reachable set R_{x_0} is dense in $\mathcal{X}$. Finally, the system is* approachable *if*

$$\mathsf{closure}\ (R_x) = \mathcal{X},\ \forall x \in \mathcal{X}.$$

When R_x is nowhere dense we will say that the reachable set is *discrete*. The term *dense in a subset $\mathcal{X}' \subset \mathcal{X}$* will be used to indicate that

$$\mathsf{closure}\ (R_x) \cap \mathcal{X}' = \mathcal{X}',\ \forall x \in \mathcal{X},$$

In practical applications, it may be important to measure the coarseness of discrete reachable sets. We will say that a configuration x_f is *ϵ–approachable* from x_0 if $\exists t \in \mathcal{T}, \omega \in \Omega$, such that $d(\mathcal{A}_{t,\omega}(x_0), x_f) < \epsilon$. The set of configurations that are ϵ–approachable from x is denoted by R_x^ϵ. The system will be said ϵ–approachable if $R_x^\epsilon = \mathcal{X}$, $\forall x \in \mathcal{X}$.

Let us consider quantized, time independent, control systems in discrete time in the form

$$x^+ = g(u, x),\ u \in \mathcal{U}, \tag{2}$$

where $x \in \mathcal{X}$, $\mathcal{X}$ a manifold, and $\mathcal{U} \subset \mathbb{R}^m$ a quantized control set. By "quantized control set" we mean sets that are finite, or that belong to a regular mesh, or to a union of a finite number of regular meshes. A quantized control set is symmetric if $w \in \mathcal{U} \Rightarrow -w \in \mathcal{U}$. Examples of symmetric quantized control sets are as follows:

$$\begin{aligned}
\mathcal{U}_1 &= \left\{\pm\begin{bmatrix}1\\0\end{bmatrix}, \pm\begin{bmatrix}0\\1\end{bmatrix}\right\}\\
\mathcal{U}_2 &= \left\{\pm\begin{bmatrix}\sqrt(2)\\1\end{bmatrix}\right\}\\
\mathcal{U}_3 &= \left\{\pm\begin{bmatrix}1\\\sqrt(3)\end{bmatrix}\right\}\\
\mathcal{U}_4 &= \left\{\pm\begin{bmatrix}1\\0\end{bmatrix}, \pm\begin{bmatrix}0\\1\end{bmatrix}, \pm\begin{bmatrix}\sqrt(2)\\1\end{bmatrix}\right\}\\
\mathcal{U}_5 &= \left\{\pm\begin{bmatrix}1\\0\end{bmatrix}, \pm\begin{bmatrix}0\\1\end{bmatrix}, \pm\begin{bmatrix}\sqrt(2)\\1\end{bmatrix}, \pm\begin{bmatrix}1\\\sqrt(3)\end{bmatrix}\right\}\\
\mathcal{U}_6 &= \left\{\pm k\begin{bmatrix}\sqrt(2)\\1\end{bmatrix}, k \in \mathbb{Z}\right\}
\end{aligned} \tag{3}$$

A formal definition of quantized control sets is now given, whose technical construction will turn out to be useful later in theorems 8 and 9.

Definition 4: A quantized control set $\mathcal{U} \subset \mathbb{R}^m$, $\mathcal{U} = \cup_{i=1}^M \mathcal{W}_i$ is a finite union of (sub)sets that can be finitely generated by linearly independent vectors. Each $\mathcal{W}_i$ is described by a triple (W_i, λ_i, S_i), with $W_i \in \mathbb{Q}^{m\times m}$ an invertible matrix, $\lambda_i \in \mathbb{R}^m$, and $S_i \subset \mathbb{Z}^m$ of cardinality c_i (possibly $c_i = \infty$), as

$$\mathcal{W}_i = \{\ \mathsf{diag}\ (\lambda_i) W_i s, s \in S_i\}.$$

In terms of such definition, the examples above are described as

$$\begin{aligned}
\mathcal{U}_1 &= \mathcal{W}_1 = (W, \lambda_1, S_1)\\
\mathcal{U}_2 &= \mathcal{W}_2 = (W, \lambda_2, S_2)\\
\mathcal{U}_3 &= \mathcal{W}_3 = (W, \lambda_3, S_2)\\
\mathcal{U}_4 &= \mathcal{W}_1 \cup \mathcal{W}_2\\
\mathcal{U}_5 &= \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3\\
\mathcal{U}_6 &= \mathcal{W}_6 = (W, \lambda_2, S_3)
\end{aligned}$$

with $W = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$, $\lambda_1 = [1, 1]$, $\lambda_2 = \left[\sqrt{2}, 1\right]$, $\lambda_3 = \left[1, \sqrt{3}\right]$, $S_1 = \{\pm\begin{bmatrix}0\\1\end{bmatrix}, \pm\begin{bmatrix}1\\-1\end{bmatrix}\}$, $S_2 = \{\pm\begin{bmatrix}1\\0\end{bmatrix}\}$, $S_3 = \{\begin{bmatrix}0\\1\end{bmatrix} a,\ a \in \mathbb{Z}\}$.

Without loss of generality, we can assume that in definition 4 vectors $\lambda \in \mathbb{R}^m$ have no null components ($\lambda_{i,k} \neq 0$, $i = 1, \ldots, M$, $k = 1, \ldots, m$). Indeed, any input subset $\mathcal{W}_i = (W_i, \lambda_i, S_i)$ with say $\lambda_{i,k} = 0$ can be rewritten as $\mathcal{W}_i = (W_i, \lambda_i', S_i')$, with $\lambda_{i,k}' \neq 0$ and S_i' suitably chosen in the nullspace of the k–th row of W_i. Also, in full generality, we can assume that different λ are not rationally related, in the sense that $\forall i, k \neq i$, $\exists j : \frac{\lambda_{i,j}}{\lambda_{k,j}} \notin \mathbb{Q}$. Indeed, if $\mathcal{W}_i = (W_i, \lambda_i, S_i)$ and $\mathcal{W}_k = (W_k, \lambda_k, S_k)$ are given such that λ_i and λ_k are rationally related, then there exist $\lambda \in \mathbb{R}^m$, $W \in \mathbb{Q}^{n\times n}$, $A_i, A_k \in \mathbb{Z}^{n\times n}$ and diagonal integer matrices M_i, M_k such that $\mathsf{diag}\ (\lambda_i) = M_i diag(\lambda)$, $\mathsf{diag}\ (\lambda_k) = M_k diag(\lambda)$, $W_i = WA_i$, and $W_k = WA_k$, so that we can have the same input set described by $\mathcal{W} = (W, \lambda, S)$ with $S = M_i A_i S_i \cup M_k A_k S_k$.

Hence, each control subset $\mathcal{W}_i \subset \mathbb{R}^m$ is comprised of points belonging to a lattice (recall that a lattice in $\mathbb{R}^m$ is an additive group which can be integrally generated by m independent vectors), and different control subsets have no common underlying lattice. In theorems 8 and 9, we will show that each discrete control set $\mathcal{W}_i$ produces, under the considered state transition maps, a lattice of reachable points whose mesh depends on λ_i. It will also turn out that if two lattices, R_0 and R_0' of reachable points from the origin arise, also every point $x_1 + x_2$, $x_1 \in R_0$, $x_2 \in R_0'$ is reachable. Hence, if two discrete control subsets are available that are not rationally related, then the whole reachable set from the origin is dense (or dense in a subset of $\mathcal{X}$).

Remark 1: Notice that, in full generality, we consider input sets that may contain irrational numbers. In most practical applications, actual occurrence of irrational quantities is impossible, because of either the use of digital equipment, or of finite modeling accuracy. However, our taking in consideration input sets with irrationally related quantities will be useful to describe limit behaviours of a system as the representation of irrational quantities gets finer and finer: this will allow for instance to study the effects of increasing machine precision in digital controllers, or those of reducing tolerances in model descriptions (as e.g. in the rolling polyhedron example with regard to the measures of edge lengths or angles). Thus, a practically important consequence of showing that the reachable set of a system under a given set of controls is dense will be that the system can be made ϵ-approachable for arbitrarily small ϵ, provided that fine enough a number representation, or a modeling tolerance, of the input set is available. This result is stated precisely in Corollary 1.

For simplicity, we will henceforth assume Ω to be comprised of all strings of symbols in $\mathcal{U}$. Obviously, such definition is equivalent to assigning a countable number of maps $\mathcal{A}_u : \mathcal{X} \to \mathcal{X}$. In this case the reachable set from a point $x \in \mathcal{X}$ is $R_x = \{\mathcal{A}_{u_1} \cdots \mathcal{A}_{u_n}(x) : n \in \mathbb{N}_0, u_i \in \mathcal{U}\}$ ($\mathbb{N}_0$ includes the number 0 so that $x \in R_x$). Moreover, we introduce the relation $\sim$ over the elements of $\mathcal{X}$ by setting $x \sim y$, $x, y \in \mathcal{X}$, if $y \in R_x$.

Quantized control systems may exhibit many peculiar phenomena with respect their differentiable counterparts, as illustrated in the next two examples.

Example 2. Consider the linear driftless system

$$x^+ = x + u \tag{4}$$

with $x \in \mathbb{R}^n$ and $u \in \mathcal{U}$, $\mathcal{U}$ quantized. For $n = 1$ and $\mathcal{U} = \{\sqrt{2}, -1\}$, $0 \sim \sqrt{2}$ but, since $\sqrt{2}$ is irrational, $\sqrt{2} \not\sim 0$. $\triangleleft$

In some of the analysis to follow, we will focus on a special class of systems that rule out this type of behavior:

Definition 5: The system (2) is said to be invertible if for every $x \in \mathcal{X}$ and $u \in \mathcal{U}$ there exists a finite sequence of controls $u_i \in \mathcal{U}$, $i = 1, \ldots, n$, such that $\mathcal{A}_{u_1} \cdots \mathcal{A}_{u_n}(\mathcal{A}(u, x)) = x$.

Obviously, the relation $\sim$ is an equivalence relation if and only if the system is invertible. For invertible systems we can partition the state space into a family of reachable sets, by taking the quotient $\mathcal{X}/\sim$ with respect to the equivalence relation $\sim$. We call the set $\widetilde{\mathcal{X}} = \mathcal{X}/\sim$ the reachability set of the system (2) and endow $\widetilde{\mathcal{X}}$ with the quotient topology, that is the largest topology such that $\pi : \mathcal{X} \to \widetilde{\mathcal{X}}$, the canonical projection, is continuous. For instance, the system of Example 2 with $\mathcal{U} = \{0, 1/2, -1\}$ is invertible. The reachable set from the origin R_0 is the subgroup of $\mathbb{R}$ generated by $1/2$ and the reachability set $\widetilde{\mathcal{X}}$ is homeomorphic to S^1.

Example 3. Consider the system

$$x^+ = g(x, u)$$

where $x \in \mathbb{R}$, $\mathcal{U} = \{\pm 1/2, \pm 2\}$ and $g(x, u) = u \cdot x$. The system is invertible, $R_0 = \{0\}$ and for every $x \neq 0$ $R_x = \{\pm 2^i x : i \in \mathbb{Z}\}$. The reachability set $\widetilde{\mathcal{X}}$ is homeomorphic to the set $S^1 \cup \{\alpha\}$, where on S^1 there is the usual topology while the only neighborhood of α is the whole space. $\triangleleft$

Notice that in example 3, the reachable set R_x for $x \neq 0$ has only one accumulation point, namely 0. More regular structures of the reachable set are obtained if we assume that $\mathcal{X}$ is a metric space and that the maps $x \mapsto g(x, u)$ are isometries. Indeed, in this case we have a dichotomy illustrated by the next proposition:

Theorem 1: Consider an invertible system (2). Let $(\mathcal{X}, d)$ be a metric space and assume that $x \to g(x, u)$ is an isometry for every $u \in \mathcal{U}$. Then, for all $x \in \mathcal{X}$, the reachable set R_x is comprised either only of accumulation points or only of isolated points.

Proof: Assume that the set R_x admits an accumulation point $\bar{x} \in R_x$. Let $x_k \in R_x$, $k \in \mathbb{Z}$ be a sequence converging to $\bar{x}$ and that the set $\{x_k : k \in \mathbb{Z}\}$ is infinite. Since the system is invertible, for every k there exists $\tilde{u}_k = (u_k^1, \ldots, u_k^{n_k})$ such that $u_k^i \in \mathcal{U}$ and $g_{u_k^1} \cdots g_{u_k^{n_k}}(x_k) = x$. Define $y_k = \lim_m g_{u_k^1} \cdots g_{u_k^{n_k}}(x_m)$. For every k and m we have:

$$\begin{aligned} &d(g_{u_k^1} \cdots g_{u_k^{n_k}}(x_m), x) = \\ &\quad d(g_{u_k^1} \cdots g_{u_k^{n_k}}(x_m), g_{u_k^1} \cdots g_{u_k^{n_k}}(x_k)) = \\ &\quad d(x_m, x_k). \end{aligned}$$

Passing to the limit in m, we have $d(y_k, x) = d(\bar{x}, x_k)$. Clearly the sequence y_k converge to x and contains infinitely many distinct points, so x is an accumulation point for R_x. Now it easily follows that all points of R_x are accumulation points for R_x. ∎

Example 2-b. The system in (4) is an interesting special case (indeed, it will turn out to play a crucial role in our treatment). It is clear that for every $x_0 \in \mathbb{R}^n$ the reachable set R_{x_0} from x_0 is equal to $x_0 + R_0$ where R_0 is the reachable set from the origin. The hypothesis of the above theorem are satisfied. Notice that if $n = 1$ and $\mathcal{U}$ is symmetric then the set R_0 is either everywhere dense or nowhere dense in $\mathbb{R}$ (since it is a subgroup of $\mathbb{R}$), hence presenting a stronger dichotomy of the one illustrated by the above theorem.

For $n > 1$ we may have more varied structures. For instance, for $n = 2$ and with reference to (3), the reachable set for the control set $\mathcal{U}_1$ is the unit lattice in $\mathbb{R}^2$. The control sets $\mathcal{U}_2$ and $\mathcal{U}_3$ provide lattices that are embedded in a one–dimensional linear manifold, while for $\mathcal{U}_5$ the reachable set is everywhere dense (see theorem 6 below). The reachable set for the infinite set $\mathcal{U}_6$ coincides with that for $\mathcal{U}_2$. As for the reachable set for $\mathcal{U}_4$, there are directions along which the set is dense and directions along which it is discrete. Indeed every subgroup G of $\mathbb{R}^n$ can be written as a direct sum $G = G_1 + G_2$ with G_1 dense in some subspace of dimension r and G_2 a lattice of rank s with $r + s = n$. Notice that if we define $\pi_v : \mathbb{R}^n \to \mathbb{R}$ to be the orthogonal projection on the direction of the vector v, then $\pi_v(R_0)$ is dense in $\mathbb{R}$ for every v not parallel to $(0, 1)$ (and this corresponds to the fact that the projection of the reachable set is precisely the reachable set of the projection of the system). On the other side, $R_0 \cap \{\lambda v : \lambda \in \mathbb{R}\}$ is discrete for every v not parallel to $(1, 0)$. $\triangleleft$

III. Linear Quantized Control Systems

In this section, we report some results on systems of the form

$$x^+ = Ax + Bu, \; u \in \mathcal{U} \tag{5}$$

with $\mathcal{U}$ a quantized set as usual, and (A, B) a controllable pair. Reachability questions that may be asked about such system can be divided in two types:

Definition 6:

Q1 given a pair (A, B), find conditions under which a quantized control set $\mathcal{U}$ exists such that the reachable set R_0 from 0 is dense in $\mathbb{R}^n$. If possible, find such a $\mathcal{U}$.
Q2 given a pair (A, B), a quantized set $\mathcal{U}$, and initial conditions x_0, determine whether or not the corresponding reachable set is dense.
We will refer to question **Q1** as to a synthesis problem, and to **Q2** as to an analysis problem.

A. Synthesis

The synthesis problem has been extensively studied in [9]. Main results are reported below.

Theorem 2 ([9]) Necessary and sufficient conditions for a quantized control set $\mathcal{U}$ to exist such that the reachable set R_0 from 0 of (5) is dense in $\mathbb{R}^n$ are that
1. (A, B) is controllable;
2. if λ is an eigenvalue of A, then $|\lambda| \geq 1$.

Remark 2: The necessity of the first condition is obvious. If the second condition does not hold, the reachable set is bounded in some component. However, a similar density result can still be obtained (provided that no eigenvalue of A is zero) if local approachability at the origin is considered instead.
Conditions for a positive answer to the synthesis problem are very weak. Proofs given in [9], though far from trivial, are constructive, as they provide explicitly a *standard* control set $\mathcal{U} = \{0, \pm u_1, \pm u_2, \ldots\}$ that achieves density for a fixed system. Furthermore, results are shown to be uniform with respect to both initial conditions and eigenvalue locations.

A further twist to the synthesis problem results from restricting control values to be rational numbers, as is natural in digital control. In particular, in applications involving uniform quantization (e.g. due to D/A conversion), inputs will be restricted as $\mathcal{U} \subset \mathbb{Q}^m$. For this case we immediately have the following "negative" synthesis result:

Theorem 3: Consider the system (5) and assume that A, B have integer entries. Then, for any $\mathcal{U} \subset \mathbb{Q}^m$, the reachable set R_0 is a subset of a lattice.

In general, if we allow the control set $\mathcal{U} \subset \mathbb{Q}^m$ to be discrete but infinite then (unless we are in the situation of the above theorem with (A, B) rational) we expect density of R_0 to be generic. The situation is profoundly different if we consider finite control sets $\mathcal{U}$, even without uniform bound on the cardinality. There is a special class of algebraic numbers that play a key role. We recall that an algebraic number λ is a real number that is root of a polynomial P with integer coefficients. If, moreover, the leading coefficient of P is 1 then λ is called an algebraic integer. For an algebraic number λ we can determine the minimal polynomial P_λ that is the polynomial of minimal degree such that $P_\lambda(\lambda) = 0$, moreover if λ is an algebraic integer P_λ can be chosen with leading coefficient 1. Given an algebraic number λ we call the other roots of P_λ the Galois conjugates of λ (obviousy they may be not real).

Definition 7: An algebraic integer $\lambda > 1$ is a Pisot number if all its Galois conjugates have modulus strictly less than one.

The following theorem holds

Theorem 4 ([9]) Consider a system (5) satisfying the assumptions of theorem 2 (necessary for density) and assume that A is in Jordan form with real eigenvalues, $B = I$ (the identity matrix). For every finite set $\mathcal{U} \subset \mathbb{Q}^n$ the reachable set R_0 is not dense in $\mathbb{R}^n$ if and only if there exists an eigenvalue of A whose modulus is a Pisot number.

Notice the strength of the theorem implying that in the case in which an eigenvalue is a Pisot number, then whatever choice of a finite set $\mathcal{U} \subset \mathbb{Q}^n$ with arbitrarily large finite cardinality gives a nondense reachable set R_0. The set of Pisot number is obviously countable but the surprising fact is that it is closed. Hence, it is not dense in $\mathbb{R}$ and indeed is "small" in a topological sense. Many facts are indeed known about the set T of Pisot numbers. For example T admits a minimum value $\lambda \sim 1.33$, that is the unique positive root of $x^3 - x - 1$. The smallest accumulation point of T is the well known golden number $(1+\sqrt{5})/2$ that is root of $x^2 - x - 1$. We refer the reader to [9] and references therein for information about Pisot numbers.

On the other hand, if all eigenvalues are not Pisot then it is possible to obtain density of R_0 choosing a large enough number M (of the order of the modulus of the biggest eigenvalue) and all controls with integer coordinates in $[-M, M]$. See [25] and [26].

B. Analysis

The analysis question is indeed much more difficult to answer. To understand the difficulty we refer the reader to [27] where the so called $\{0, 1, 3\}$-problem is studied. This corresponds exactly to the analysis of the Hausdorff measure of the reachable set for the system $x^+ = \lambda x + u$, $x \in \mathbb{R}$, $0 < \lambda < 1$, $u \in \mathcal{U} = \{0, 1, 3\}$, if we allow infinite sequences of controls. The analysis problem has some partial answer in the cited paper and references therein.

Another strictly linked number theory problem is the one considered in [25]. We refer the reader to [9] for a deeper discussion of the links between these hard mathematical problems. From the results of [25] it is even more clear the role played by Pisot numbers.

In this section, we provide some results on the analysis question concerning the simple but fundamental case of driftless linear systems

$$x^+ = x + u \tag{6}$$

where $x \in \mathbb{R}^n$ and u takes values in a quantized set $\mathcal{U} \subset \mathbb{R}$.

Our aim is to find conditions for the reachable set from any initial point to be dense in $\mathbb{R}^n$, or otherwise study its structure. To do so, we start by considering system (6) with $n = 1$.

Given two real numbers r_1, r_2, we write $r_1 \cong r_2$ when $\frac{r_1}{r_2} \in \mathbb{Q}$. Obviously $\cong$ is an equivalence relation. Consider the condition
(C) There exist $u, v \in \mathcal{U}$ such that $u \not\cong v$ and there exist $u', v' \in \mathcal{U}$ such that $u' \cdot v' < 0$
and notice that it is equivalent to
(C') There exist $u, v \in \mathcal{U}$ such that $u \not\cong v$ and $u \cdot v < 0$.

Indeed, obviously (C') implies (C). On the other hand, assume that (C) is true, then $\mathcal{U}^{\pm}=\mathcal{U}\cap\mathbb{R}^{\pm}$ are nonempty. If for every $u\in\mathcal{U}^{+}$ and $v\in\mathcal{U}^{-}$ we have $u\cong v$ then, since $\cong$ is an equivalence relation we get that all control have rational ratio, a contradiction.

We start reporting the following result

Lemma 1 ([9]) The reachable set R_0 from the origin for system (6) with $n=1$ is dense if and only if there exist two sequences $c_k\in R_0$ and $d_k\in R_0$ both converging to zero such that $d_k<0<c_k$.

Let us now prove the following

Theorem 5: Let R_0 be a reachable set for the system (6) with $n=1$ from the origin. Then R_0 is dense if and only if (C) holds true. Moreover, if R_0 is not dense then it is nowhere dense.

Proof: Let us first assume that (C) holds true and let $u,v\in\mathcal{U}$ be as in (C'). Since the ratio $\frac{u}{v}$ is not rational we can consider the sequence $\frac{p_k}{q_k}\in\mathbb{Q}$, p_k, q_k integers, $q_k>0$, given by its continued fraction. We have:

$$\frac{u}{v}-\frac{p_k}{q_k}=(-1)^k\varepsilon_k$$

where $0<\varepsilon_k<\frac{1}{q_k^2}$ and q_k grows to infinity. We get immediately:

$$q_k\,u+(-p_k)v=(-1)^k v\varepsilon_k q_k.$$

From $u\cdot v<0$ we get $-p_k>0$, hence $q_k\,u+(-p_k)v\in R_0$. Now the required sequences are obtained setting, if $v>0$, $c_k=q_k\,u+(-p_k)v$ for k even and $d_k=q_k\,u+(-p_k)v$ for k odd and the opposite if $v<0$.

Assume now that (C') does not hold. Then either $u\cdot v>0$ for every $u,v\in\mathcal{U}$ or $u\cong v$ for every $u,v\in\mathcal{U}$. In the first case it is obvious that the set R_0 is contained either in $\mathbb{R}^+$ or in $\mathbb{R}^-$. In the latter case, the proof is as follows. Let $\mathcal{U}=\{v_1,\ldots,v_N\}$ and assume $v_1\neq 0$ then there exists p_i,q_i, such that $v_i=\frac{p_i}{q_i}v_1$. Any point of the reachable set R_{x_0} from x_0 can be written as x_0+a, $a=m_1\,v_1+\ldots+m_N\,v_N$ with $m_i\in\mathbb{N}$. Thus:

$$\begin{aligned} a &= m_1\,v_1+\ldots+m_N\,v_N=v_1\sum_{i=1}^{N}\frac{m_i\,p_i}{q_i}\\ &= v_1\left(\frac{\sum_{i=1}^{n}m_i\,p_i\,q_1\cdots q_{i-1}q_{i+1}\cdots q_n}{q_1\cdots q_n}\right).\end{aligned}$$

Now if $a\neq 0$ we have that the numerator of the above expression is different from zero and being an integer is at least of modulus 1. Therefore, if $a\neq 0$ we get

$$|a|\geq\frac{|v_1|}{|q_1\cdots q_n|}$$

and obviously R_0 can not be dense. Moreover, from the same expression we have that a is always a multiple of $v_1/(q_1\cdots q_n)$ hence R_0 is indeed nowhere dense. ∎

Since the reachable set from a point x_0 is exactly x_0+R_0, we have a dichotomy similar to that of Section II, even though, for asymmetric sets $\mathcal{U}$, R_0 may fail to be a subgroup of $\mathbb{R}$. Next, let us consider the system (6) with $x\in\mathbb{R}^n$.

Theorem 6: For the set R_0 of configurations reachable from the origin for system (6), the following hold:

i) A necessary condition for the reachable set R_0 to be dense is that $\mathcal{U}$ contains $n+1$ controls of which n are linearly independent;

ii) If $\mathcal{U}=\{v_1,\ldots,v_{n+1}\}$, whereof $v_1,\ldots,v_n$ are linearly independent, and w_i are the components of v_{n+1} w.r.t. to the other v_i's, then R_0 is dense if and only if w_i is negative for all i and $1,w_1,\ldots w_n$ are linearly independent over $\mathbb{Z}$, that is $a_0+a_1w_1+\cdots+a_nw_n=0$, $a_i\in\mathbb{Z}$, if and only if $a_i=0$ for all i;

iii) If $u_1,\ldots,u_n\in\mathcal{U}$ are linearly independent and there exists n irrational negative numbers $\alpha_1,\ldots,\alpha_n$ such that $v_i=\alpha_iu_i\in\mathcal{U}$ for every $i=1,\ldots,n$ then R_0 is dense in $\mathbb{R}^n$;

iv) If there exist m linearly independent vectors $v_i\in\mathbb{Q}^n$ such that $\forall u\in\mathcal{U}$, there exist m integers $a_1,\ldots,a_m$ such that $u=\sum_i a_iv_i$, then R_0 is discrete (actually, a subset of a lattice) in $\mathbb{R}^n$.

Proof: While i) and iv) are obvious, iii) follows directly from application of arguments used in the proof of theorem 5. We now prove ii).

Assume first that w_i are negative and that the linear diophantine equation

$$a_0+a_1w_1+\cdots+a_nw_n=0$$

has unique integer solution $a_i=0$, $i=0,\ldots,n$. Given $r\in\mathbb{R}$, we indicate by $[r]$ its integer part and by $(r)=r-[r]$ its fractional part. By Kroneker's theorem (see e.g. [28]), the sequence $\{((mw_1),\ldots,(mw_n)) : m\in\mathbb{N}\}$ is dense in the unit n-cube. Take $v\in\mathbb{R}^n$ and let λ_i be its coordinates w.r.t. the basis $\{v_1,\ldots,v_n\}$. For every $\varepsilon>0$ there exists $m\in\mathbb{N}$ such that $|[mw_i]|>|[\lambda_i]|$ and $|(\lambda_i)-(mw_i)|<\varepsilon$. Hence, $|v-mv_{n+1}-\sum_i([\lambda_i]-[mw_i])v_i|<\varepsilon\,(\sum_i|v_i|)$. Since w_i are negative so are $[mw_i]$, and from the choice of m we get that $mv_{n+1}+\sum_i([\lambda_i]-[mw_i])v_i\in R_0$. Since ε is arbitrary, we conclude one implication.

If some w_i is positive then clearly the projection of any $x\in R_0$ along v_i is positive. Assume that there exist integers λ_i, $i=1,\ldots,n+1$, not all vanishing, such that $\sum_{i=1}^n\lambda_iw_i+\lambda_{n+1}=0$ and, with no loss of generality, that $v_i=e_i$, where $\{e_i,\ i=1,\ldots,n\}$ is the canonical base of $\mathbb{R}^n$. Given $x=\sum_i\mu_iv_i+\lambda v_{n+1}\in R_0$, $\mu_i\in\mathbb{N}$, $\lambda\in\mathbb{N}$, we have that $x\cdot(1,\ldots,1)=\sum_i\mu_i+\lambda\sum_i w_i$ is a discrete subset of $\mathbb{R}$, thus R_0 is not dense. ∎

Necessary and sufficient conditions for approachability can be given under stronger hypotheses on the control set.

Definition 8: A quantized control set $\mathcal{U}=\cup_{i=1}^{M}\mathcal{W}_i\subset\mathbb{R}^m$ with $\mathcal{W}_i=(W_i,\lambda_i,S_i)$ as in Definition 4, is a *regular* control set if

- it is symmetric;
- each set $\mathcal{W}_i$ contains m linearly independent vectors.

Moreover, we say that the quantized control set is *sufficiently rich* if the following holds. For all $i=1,\ldots,M$, $\mathcal{W}_i$ contains c_i' vectors with $m+1\leq c_i'<\infty$, pairwise not parallel and m of which are linearly independent. All the other vectors of $\mathcal{W}_i$ are parallel to some of these c_i' vectors.

Theorem 7: Let the set $\{W_is, s\in S_i\}$ be symmetric. The reachable set of $x^+=x+u$, $x\in\mathbb{R}^m$, $u\in\{W_is, s\in S_i\}$

is a lattice generated by m linearly independent vectors if and only if $\{W_i s, s \in S_i\}$ contains m linearly independent vectors.

Proof: The necessity part is obvious. We prove the sufficiency part. By definition each element of $\{W_i s, s \in S_i\}$ is written as an integer combination of m linearly indepedent vectors (the columns of W_i) of $\mathbb{Q}^m$, then by theorem 6 *iv)* we have that the reachable set is discrete. Recalling that the set contains m linearly independent vectors and it is symmetric we conclude that the reachable set is a subgroup of $\mathbb{R}^m$, hence it is an m–dimensional lattice. ∎

In the following we will denote $\{\bar{w}_{i,1}, \ldots, \bar{w}_{i,m}\}$ the basis for the m–dimensional lattice generated by the set $\{W_i s, s \in S_i\}$ which satisfy the hypothesis of theorem 7.

Theorem 8: Consider the system $x^+ = x + u$, $x \in \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^n$ with $\mathcal{U} = \cup_{i=1}^{M} \mathcal{W}_i$ a regular quantized control set. Then we have the following cases

1. If $M = 1$, the reachable set is a lattice with basis $\{\mathsf{diag}\,(\lambda_1)\bar{w}_{1,j}; j = 1, \ldots, n\}$;

2. If $M \geq 2$, for every $j = 1, \ldots, n$ consider the corresponding condition

$$[\mathbf{C.j}]\ \exists\ i, k \in \{1, \ldots, M\} \text{ such that } \frac{\lambda_{i,j}}{\lambda_{k,j}} \notin \mathbb{Q}$$

Then we have:

2.a if $\mathbf{C.j}$ holds for $j = 1, \ldots, n$, then the reachable set is dense in $\mathbb{R}^n$;

2.b otherwise, let $J \subset \{1, \ldots, n\}$ be such that $\mathbf{C.j}$ holds iff $j \in J$, and let $|J|$ denote the cardinality of J. Then there exists a subspace $V \subset \mathbb{R}^n$ of dimension $|J|$ and a lattice $L \subset \mathbb{R}^n$ generated by $n - j$ vectors (not belonging to V) such that the reachable set is dense in $L + V$.

Proof: The first part of the thesis is obvious. As for part 2.a, denote $\bar{W}_i$ the matrix of columns $\bar{w}_{i,1}, \ldots, \bar{w}_{i,n}$. Without loss of generality we can assume that $\forall i = 1, \ldots, M$ has integer entries. For every vector $v \in \mathsf{det}\,(\bar{W}_i)\mathbb{Z}^n$ we can solve the system $\bar{W}_i x = v$ for $x \in \mathbb{Z}^n$. Hence for every $v_i \in \mathsf{det}\,(\bar{W}_i)\mathbb{Z}^n$, $i = 1, \ldots, M$ we can reach the point

$$\mathsf{diag}\,(\lambda_1)v_1 + \cdots + \mathsf{diag}\,(\lambda_M)v_M.$$

By conditions $\mathbf{C.j}$ for $j = 1, \ldots, n$ these points form a dense set in $\mathbb{R}^n$. The proof of 2.b can be obtained in the same way. ∎

IV. Nonlinear Driftless Systems

As already pointed out, quantized control systems pose nontrivial problems, particularly from the analysis point of view. Such problems are even more severe with nonlinear systems. However, it turns out that for a particular, yet important class of systems, the analysis problem can be given a complete solution. Consider the discrete–time analog of the much studied class of continuous–time, driftless, nonholonomic systems that can be written in chained form ([10])

$$\begin{array}{rcl} \dot{x}_1 &=& u_1 \\ \dot{x}_2 &=& u_2 \\ \dot{x}_3 &=& x_2 u_1 \\ \vdots &=& \vdots \\ \dot{x}_n &=& x_{n-1} u_1 \end{array} . \qquad (7)$$

Consider now the discrete system

$$\begin{array}{rcl} x_1^+ &=& x_1 + u_1 \\ x_2^+ &=& x_2 + u_2 \\ x_3^+ &=& x_3 + x_2 u_1 + u_1 u_2 \frac{1}{2} \\ x_4^+ &=& x_4 + x_3 u_1 + x_2 u_1^2/2 + u_1^2 u_2 \frac{1}{6} \\ \vdots &=& \vdots \\ x_n^+ &=& x_n + \sum_{j=1}^{n-2} x_{n-j} \frac{u_1^j}{j!} + u_1^{n-2} u_2 \frac{1}{(n-1)!}, \end{array} \qquad (8)$$

which can be regarded as system (7) under unit sampling. We are interested in studying the reachability set of system (8) with $(u_1, u_2) \in \mathcal{U} \subset \mathbb{R}^2$, a quantized control set. System (8) is invertible While this property will be proved in the sequel (see section IV-A), they can be expected from the fact that system (8) is an exact sampled model of system (7), and should hence inherit such property (on the opposite, a discrete-time approximation of (7) such as that obtained by the forward Euler method would not be invertible).

In order to study the reachability set of system (8), our program is to show first that the reachability analysis in the whole state space $\mathbb{R}^n$ can be decoupled in the reachability analysis in the "base" space spanned by the first two variables (x_1, x_2), and in the "fiber" space $(x_3, \ldots, x_n)$ corresponding to a given reachable base point, $(\bar{x}_1, \bar{x}_2)$ (such base-fiber decomposition of state space is standard in the nonholonomic literature, see e.g. [29] and [30]). Reachability in the base space will then be studied by results reported in the previous section, and the rest of the paper will be devoted to the study of reachability in the fiber space.

The summarizing result of our reachability analysis for chained–form systems under unit sampling with quantized control is stated below:

Theorem 9: Consider system (8) with controls belonging to a regular and sufficiently rich quantized control set $\mathcal{U}$ as in Definitions 4 and 8. Then we have the following cases:

1) if $M = 1$, the reachable set is a lattice;

2) if $M \geq 2$ and both conditions $\mathbf{C.1}$ and $\mathbf{C.2}$ in theorem 8 hold, the reachable set is dense in the state space;

3) if $M \geq 2$ and either condition $\mathbf{C.1}$ or $\mathbf{C.2}$ in theorem 8 does not hold, there exists a subspace V of dimension $n-1$ and a lattice L generated by a single vector $\ell \notin V$ such that the reachable set is contained and dense in $L + V$.

The proof of these results, which is reported in section IV-B, is constructive. For cases where the reachable set is a lattice, we provide in lemma 8 explicitly a finite set of generators, such that steering on the lattice is reduced to solving a linear Diophantine equation, which can be done in polynomial time (see e.g. [31]). If the reachable set is dense the problem of steering the state to an ϵ neighborhood of a desired point, that is to have ϵ–approachability, can be solved by constructing, as shown in Corollary 1, lattice approximations of the reachable set with sufficient granularity. The case where the reachable set is dense in a subset of the state space is analogous, provided that the desired final point belongs to the closure of the reachable set.

A. Invertibility of quantized chained form systems

Consider system (8) with a symmetric set of input symbols $\mathcal{U}$. The set of input words $\Omega = \{\text{strings of symbols in } \mathcal{U}\}$ is a group for string concatenation, with the relation $(-v)v = v(-v) = \emptyset$ (empty string) and inverse

$$(v_1 v_2 \cdots v_m)^{-1} = -v_m \cdots - v_2 - v_1,$$

$\pm v_i \in \mathcal{U}, \forall i$. In full generality, the state-transition map for system (8) can be written as

$$\mathcal{A}(\omega, x) = x + A(\omega, x) + \Delta(\omega). \tag{9}$$

For an input word with N symbols, $\omega = v_1 v_2 \cdots v_N$, denoting by $v_{i,j}$ the j–th component of v_i, by simple calculations one finds for the first two components $A_1(\omega, x) = A_2(\omega, x) = 0$ and

$$\Delta_1(\omega) = \sigma = \sum_{i=1}^{N} v_{i,1},$$

$$\Delta_2(\omega) = \tau = \sum_{i=1}^{N} v_{i,2},$$

Moreover, introducing the shorthand notation

$$\sigma_i = \sigma_i(\omega) = \begin{cases} \sum_{j>i}^{N} v_{j,1} & \text{if } i < N \\ 0 & \text{if } i = N \end{cases}$$

we have:

Lemma 2: The addends of $\mathcal{A}(\omega, x)$ can be written as:

$$A_j(\omega, x) = \sum_{r=2}^{j-1} \frac{1}{(j-r)!} x_r \sigma^{j-r}, \quad j \geq 3,$$

and

$$\Delta_j(\omega) = \frac{1}{(j-1)!} \sum_{i=1}^{N} \frac{v_{i,2}}{v_{i,1}} \left((v_{i,1} + \sigma_i)^{j-1} - \sigma_i^{j-1} \right).$$

The proof is given in the appendix.

Using the above expression of the state–transition map, we can now prove invertibility of the system:

Theorem 10: System (8) is invertible with any symmetric control set.

Proof: It is sufficient to show that $\mathcal{A}(\omega^{-1}, \mathcal{A}(\omega, x)) = x$, with $\omega = v \in \mathcal{U}$. Immediately we have that $\Delta_1(v, -v) = \sigma = 0$ and $\Delta_2(v, -v) = \tau = 0$. From $\sigma = 0$ and lemma 2, we also get $A_j((v, -v), x) = 0$ and $\Delta_j(v, -v) = 0$, $\forall j \geq 3$. ■

B. Proof of theorem 9

Consider now the subgroup $\tilde{\Omega} \subset \Omega$ of control words that take the base variables back to their initial configuration. These are sequences of inputs such that the sum of the first and second components are zero, i.e. $\sigma = \tau = 0$. For all $\tilde{\omega} \in \tilde{\Omega}$ and $\forall x$, $A(\tilde{\omega}, x) = 0$. Hence, the action of this subgroup on the fiber is additive: $\mathcal{A}(\tilde{\omega}, x) = x + \Delta(\tilde{\omega})$. [1]

Because of additivity, $x \to \mathcal{A}(\tilde{\omega}, x)$ is an isometry (w.r.t. the Euclidean norm) for all $\tilde{\omega} \in \tilde{\Omega}$. Hence, by theorem 1 , the reachable set is comprised either of isolated points or of accumulation points. Moreover, $\mathcal{A}(\tilde{\omega}, x) = x + \Delta(\tilde{\omega})$, so that without loss of generality we may study the reachable points along the fiber over any base point, and in particular over $\bar{x}_1 = 0, \bar{x}_2 = 0$. Along any other fiber the reachable set will have the same structure, up to a translation.

System (8) can therefore be decomposed, to the purposes of reachability analysis, in two different discrete systems of the form (5). The first subsystem (which we will call "base" system), is simply $y^+ = y + u$ with $y = (x_1, x_2) \in \mathbb{R}^2$ and $u \in \mathcal{U} \subset \mathbb{R}^2$. The second (or "fiber") subsystem is given by $z^+ = z + v$ with $z = (x_3, x_4, \ldots, x_n) \in \mathbb{R}^{n-2}$ and $v \in \mathcal{V} \subset \mathbb{R}^{n-2}$ where $\mathcal{V} = \{\Delta^f(\omega),\ \omega \in \tilde{\Omega}\}$ (Δ^f denotes the $n-2$–dimensional projection of Δ on the fiber space). The control set $\mathcal{V}$ is itself symmetric. Indeed if $\omega \in \tilde{\Omega}$ then also $\omega^{-1} \in \tilde{\Omega}$ and, by the invertibility property (see theorem 10), $\Delta^f(\omega^{-1}) = -\Delta^f(\omega)$.

Observe that theorem 8 can be used in order to compute the reachable set for $y \in \mathbb{R}^2$. On the other hand, $\mathcal{V}$ is not finite, nor is it known whether it is quantized in the sense of definition 4, and hence conditions of theorems 6 and 8 cannot be checked directly.

We begin by proving case 1) of theorem 9, which we restate here for convenience.

Claim 1: The reachable set of system (8) for a sufficiently rich quantized control set $\mathcal{U} = (W, \lambda, S)$ is a lattice in $\mathbb{R}^n$.

Proof: From theorem 8, we have directly that the reachable set of the base system is a lattice generated by $\mathsf{diag}\,(\lambda)\bar{w}_1$, $\mathsf{diag}\,(\lambda)\bar{w}_2$, with $\bar{w}_1, \bar{w}_2$ a basis for the lattice generated by the elements of $\{Ws,\ s \in S\}$.

In order to analyse the structure of the reachable set of the fiber system we proceed as follows: in lemma 3 a characterization of the set $\tilde{\Omega}$ is provided, and a set C of generators for $\tilde{\Omega}$ is given in lemma 4. The translation $\Delta^f(\omega)$ with $\omega \in C$ is described in lemma 5. Then the set $\mathcal{V} = \{\Delta^f(\omega), \omega \in \tilde{\omega}\}$, which can be written as the group of translation of $\mathbb{R}^{n-2}$ generated by $\Delta_C = \{\Delta^f(\omega), \omega \in C\}$, is completely determined. To give a complete charachterization of $\mathcal{V}$ we provide, in lemma 6 a finite set B of generators for Δ_C which allow us to show that there exists λ^f such that each element in $\mathcal{V}$ can be written as $\mathsf{diag}\,(\lambda^f)v$, for some v with rational components. For applying theorem 8 with $M = 1$ and conclude that the reachable set is a lattice

[1] Notice that this represents a significant departure, and simplification, from the behavior of the continuous model (7), where the action of the generic cyclic control is additive only on the first fiber variable, x_3, and more restricted subgroups should be searched within $\tilde{\Omega}$ that have additive action on the rest of the fiber.

it will remain to give a basis in lemma 8 for the lattice of the reachable points of $z^+ = z + v$ for $v \in \mathcal{V}$ which fact, by theorem 7, is equivalent to prove that the control set $\mathcal{V}$ is regular.

As a first step, a set of generators for $\tilde{\Omega}$ is characterized.

Recall that we are assuming that $\mathcal{U} = (W, \lambda, S)$ is regular and sufficiently rich. Hence it contains $3 \le c' < \infty$, pairwise not parallel elements, of which 2 are linearly independent. In order to characterize the reachable set, it is not restrictive to assume that the cardinality of $\mathcal{U}$ is finite with $c = 2c'$ ($\mathcal{U}$ is symmetric). We can then identify S with a $2 \times 2c'$–matrix with integer coefficients such that $S = [S_+, S_-]$ where S_+ and S_- are $2 \times c'$ matrices with $S_- = -S_+$. Denote s_i the i–th column of S_+ and let $\Sigma : \Omega \to \mathbb{Z}^{c'}$ be defined for $\omega = v_1, \ldots, v_N$ as $\Sigma(\omega) = (\beta_1, \ldots, \beta_{c'})$ where $\beta_i = \sum_{j=1}^N \delta_{ij}$ and

$$\delta_{ij} = \begin{cases} 1 & if\ v_j = \ \mathsf{diag}\,(\lambda) W s_i \\ -1 & if\ v_j = -\,\mathsf{diag}\,(\lambda) W s_i \qquad i = 1, \ldots, c'. \\ 0 & \text{otherwise} \end{cases}$$

Σ counts the number of appearances of different symbols in a string, taking their signs into account.

Remark 3: For the map Σ the following properties hold
a) if $\omega_1, \omega_2 \in \Omega$ then $\Sigma(\omega_1\omega_2) = \Sigma(\omega_1) + \Sigma(\omega_2)$;
b) for all $\omega \in \Omega$, $\Sigma(\omega^{-1}) = -\Sigma(\omega)$;
c) if $\omega_1 = v_1 \ldots, v_N$ and ω_2 is obtained by permutation of symbols of ω_1, then $\Sigma(\omega_1) = \Sigma(\omega_2)$.
If ω_1 and ω_2 are as in c) then we denote $\omega_2 \equiv \omega_1$. Furthermore,
d) by a),b) and c), if $\omega_1 \equiv \omega_2$ then $\Sigma(\omega_1\omega_2^{-1}) = 0$

Let N_W denote the $c' \times (c'-2)$–matrix with integer coefficients such that $S_+ N_W = 0$, and, $\forall j = 1, \ldots, c'-2$, $G.C.D.\{(N_W)_{ij}, i = 1, \ldots, c'\} = 1$. Then we have

Lemma 3: The subgroup $\tilde{\Omega}$ can be characterized as:

$$\tilde{\Omega} = \{\omega \in \Omega | \Sigma(\omega) = (N_W \alpha),\ \alpha \in \mathbb{Z}^{c'-2}\}.$$

Proof: Let ω be such that $\Sigma(\omega) = (N_W\alpha)$ for some $\alpha \in \mathbb{Z}^{c'-2}$. Then, collecting together symbols from $\mathcal{U}$,

$$\pi_{\mathbb{R}^2}\mathcal{A}(\omega, x) = \pi_{\mathbb{R}^2}\mathcal{A}(\underbrace{z_1 \ldots z_1}_{|\beta_1| times} \cdots \underbrace{z_{c'} \ldots z_{c'}}_{|\beta_{c'}| times}, x)$$

where $\pi_{\mathbb{R}^2} : \mathbb{R}^n \to \mathbb{R}^2$ is the canonical projection on the first two components of $\mathbb{R}^n$ onto $\mathbb{R}^2$, $(\beta_1, \ldots, \beta_{c'}) = \Sigma(\omega)$ and

$$z_i = \begin{cases} \mathsf{diag}\,(\lambda) W s_i & if\ \beta_i > 0 \\ -\,\mathsf{diag}\,(\lambda) W s_i & if\ \beta_i < 0 \end{cases}$$

Recalling that $\Sigma(\omega) = (N_W\alpha)$ then $\pi_{\mathbb{R}^2}\mathcal{A}(\omega, x) = \pi_{\mathbb{R}^2}(x) + \mathsf{diag}\,(\lambda) W S_+ \Sigma(\omega) = \pi_{\mathbb{R}^2}(x) + \mathsf{diag}\,(\lambda) W S_+ N_W \alpha = \pi_{\mathbb{R}^2}(x)$. Then $\omega \in \tilde{\Omega}$.

Viceversa let $\omega \in \tilde{\Omega}$. Suppose for absurd that $S_+\Sigma(\omega) \neq 0$. Then by permuting the symbols of ω one has that

$$\omega \equiv \underbrace{z_1 \ldots z_1}_{|\beta_1| times} \cdots \underbrace{z_{c'} \ldots z_{c'}}_{|\beta_{c'}| times} = \ \mathsf{diag}\,(\lambda) W S_+ \Sigma(\omega) \neq 0$$

Then $\pi_{\mathbb{R}^2}\mathcal{A}(\omega, x) = \pi_{\mathbb{R}^2}(x) + \mathsf{diag}\,(\lambda) W S_+ \Sigma(\omega) \neq \pi_{\mathbb{R}^2}(x)$, which is a contradiction (end of proof for lemma 3). ■

Consider now the finite subset of $\tilde{\Omega}$ given by

$$\begin{aligned} \mathcal{L} \;=\; & \{\omega \in \Omega \mid \Sigma(\omega) = \pm(N_W)_j, \text{the j–th column of } N_W, \\ & \omega \text{ of minimal length }\}. \end{aligned}$$

In other terms, if $\omega \in \mathcal{L}$ contains a symbol, it does not contain its opposite.

Lemma 4:

$$C = \{\omega\tilde{\omega}\omega^{-1}; \omega \in \Omega, \tilde{\omega} \in \mathcal{L}\}$$

is a set of generators for $\tilde{\Omega}$.
The proof is given in the appendix.

Remark 4: If the control set is regular but not sufficiently rich then the set C reduces to the empty word. In this case to generate $\tilde{\Omega}$ we need to consider the commutators of words in Ω.

Lemma 5: $\forall \omega = (v_1 \cdots v_N) \in \Omega, \tilde{\omega} \in \tilde{\Omega}$

$$\Delta^f(\omega\tilde{\omega}\omega^{-1}) = G(\omega)\Delta^f(\tilde{\omega})$$

with

$$G(\omega) = \exp(-J_0\sigma)$$

where J_0 is a $(n-2)$ lower Jordan block with zero eigenvalues and $\sigma = \sigma(\omega) = \sum_{i=1}^N v_{i,1}$.

Proof: Let $\tilde{\omega} = (u_1, \ldots, u_{N_1}) \in \tilde{\Omega}$ and $\omega = (v_1 \cdots v_{N_2}) \in \Omega$. Denote $\bar{\omega} = \omega\tilde{\omega}\omega^{-1}$, then, for $j \ge 3$

$$\begin{aligned} \Delta_j(\bar{\omega}) = & \tfrac{1}{(j-1)!}\textstyle\sum_{i=1}^{N_2} \frac{v_{i,2}}{v_{i,1}} \Big((v_{i,1} + \sigma_i(\bar{\omega}))^{j-1} - \sigma_i^{j-1}(\bar{\omega})\Big) + \\ & \tfrac{1}{(j-1)!}\textstyle\sum_{i=1}^{N_1} \frac{u_{i,2}}{u_{i,1}} \Big((u_{i,1} + \sigma_{N_2+i}(\bar{\omega}))^{j-1} - \sigma_{N_2+i}^{j-1}(\bar{\omega})\Big) + \\ & \tfrac{1}{(j-1)!}\textstyle\sum_{i=1}^{N_2} \frac{v_{N_2+1-i,2}}{v_{N_2+1-i,1}} \Big((-v_{N_2+1-i,1} + \sigma_{N_1+N_2+i}(\bar{\omega}))^{j-1} \\ & \qquad\qquad -\sigma_{N_1+N_2+i}^{j-1}(\bar{\omega})\Big) \end{aligned}$$

We substitute in $\Delta_j(\bar{\omega})$ the expression for $\sigma_\ell(\bar{\omega})$ in terms of $\sigma_\ell(\tilde{\omega})$ and $\sigma_\ell(\omega)$ as follows

$$\begin{cases} \sigma_\ell(\omega) + \sigma(\tilde{\omega}) - \sigma(\omega) & \text{if } 1 \le \ell \le N_2 \\ \sigma_{\ell-N_2}(\tilde{\omega}) - \sigma(\omega) & \text{if } N_2 + 1 \le \ell \le N_2 + N_1 \\ -\sigma(\omega) + \sigma_{2N_2+N_1-\ell}(\omega) & \text{if } N_2 + N_1 + 1 \le \ell \le 2N_2 + N_1 \end{cases}$$

hence

$$\begin{aligned} & \Delta_j(\bar{\omega}) = \\ & \tfrac{1}{(j-1)!}\textstyle\sum_{i=1}^{N_2} \frac{v_{i,2}}{v_{i,1}} \Big((v_{i,1} + \sigma_i(\omega) + \sigma(\tilde{\omega}) - \sigma(\omega))^{j-1} \\ & \qquad - (\sigma_i(\omega) + \sigma(\tilde{\omega}) - \sigma(\omega))^{j-1}\Big) + \\ & \tfrac{1}{(j-1)!}\textstyle\sum_{i=1}^{N_1} \frac{u_{i,2}}{u_{i,1}} \Big((u_{i,1} + \sigma_i(\tilde{\omega}) - \sigma(\omega))^{j-1} \\ & \qquad - (\sigma_i(\tilde{\omega}) - \sigma(\omega))^{j-1}\Big) + \\ & \tfrac{1}{(j-1)!}\textstyle\sum_{i=1}^{N_2} \frac{v_{N_2+1-i,2}}{v_{N_2+1-i,1}} \Big((-v_{N_2+1-i,1} - \sigma(\omega) + \sigma_{N_2-i}(\omega))^{j-1} \\ & \qquad -(\,-\sigma(\omega) + \sigma_{N_2-i}(\omega)\,)^{j-1}\Big). \end{aligned}$$

Moreover, collecting together the first and the last sums, and recalling that $\sigma(\tilde{\omega}) = 0$ we have:

$$\begin{aligned} & \Delta_j(\bar{\omega}) = \\ & \tfrac{1}{(j-1)!}\textstyle\sum_{i=1}^{N_2} \frac{v_{i,2}}{v_{i,1}} \Big((v_{i,1} + \sigma_i(\omega) - \sigma(\omega))^{j-1} \\ & \qquad - (\sigma_i(\omega) - \sigma(\omega))^{j-1} \\ & \qquad + (-v_{i,1} - \sigma(\omega) + \sigma_{i-1}(\omega))^{j-1} \\ & \qquad -(\,-\sigma(\omega) + \sigma_{i-1}(\omega)\,)^{j-1}\Big) + \\ & \tfrac{1}{(j-1)!}\textstyle\sum_{i=1}^{N_1} \frac{u_{i,2}}{u_{i,1}} \Big((u_{i,1} + \sigma_i(\tilde{\omega}) - \sigma(\omega))^{j-1} \\ & \qquad - (\sigma_i(\tilde{\omega}) - \sigma(\omega))^{j-1}\Big). \end{aligned}$$

Finally, observing that $\sigma_{i-1}(\omega) = \sigma_i(\omega) + v_{i,1}$ we obtain:

$$\begin{aligned}&\Delta_j(\bar{\omega}) = \\ &\tfrac{1}{(j-1)!}\textstyle\sum_{i=1}^{N_1} \tfrac{u_{i,2}}{u_{i,1}} \Big((u_{i,1} + \sigma_i(\tilde{\omega}) - \sigma(\omega))^{j-1} \\ &\qquad - (\sigma_i(\tilde{\omega}) - \sigma(\omega))^{j-1} \Big).\end{aligned}$$

We rewrite the coefficient of $\frac{u_{i,2}}{u_{i,1}}$ in the sum as

$$\begin{aligned}&(u_{i,1} + \sigma_i(\tilde{\omega}) - \sigma(\omega))^{j-1} - (\sigma_i(\tilde{\omega}) - \sigma(\omega))^{j-1} = \\ &\textstyle\sum_{k=0}^{j-1} \binom{j-1}{k} (-\sigma(\omega))^{j-1-k} \Big((u_{i,1} + \sigma_i(\tilde{\omega}))^k - (\sigma_i(\tilde{\omega}))^k \Big)\end{aligned}$$

and substitute it into the expression for $\Delta_j(\bar{\omega})$:

$$\begin{aligned}&\Delta_j(\bar{\omega}) = \\ &\tfrac{1}{(j-1)!}\textstyle\sum_{i=1}^{N_1} \tfrac{u_{i,2}}{u_{i,1}} \Big(\sum_{k=0}^{j-1} \binom{j-1}{k} (-\sigma(\omega))^{j-1-k} \Big((u_{i,1} + \sigma_i(\tilde{\omega}))^k - (\sigma_i(\tilde{\omega}))^k \Big) \Big) = \\ &\textstyle\sum_{k=0}^{j-1} \tfrac{1}{(j-1-k)!} (-\sigma(\omega))^{j-1-k} \Big(\tfrac{1}{k!} \sum_{i=1}^{N_1} \tfrac{u_{i,2}}{u_{i,1}} \Big((u_{i,1} + \sigma_i(\tilde{\omega}))^k - (\sigma_i(\tilde{\omega}))^k \Big) \Big)\end{aligned}$$

Notice that the coefficient of $(-\sigma(\omega))^{j-1-k}$, is

$$\frac{1}{k!} \sum_{i=1}^{N_1} \frac{u_{i,2}}{u_{i,1}} \Big((u_{i,1} + \sigma_i(\tilde{\omega}))^k - (\sigma_i(\tilde{\omega}))^k \Big) =$$

$$\begin{cases} 0 & \text{for } \ k = 0 \\ \sum_{i=1}^{N_1} u_{i,2} = \Delta_2(\tilde{\omega}) = 0 & \text{for } \ k = 1 \\ \Delta_{k+1}(\tilde{\omega}) & \text{for } \ k > 1, \end{cases}$$

hence

$$\begin{aligned}&\Delta_j(\bar{\omega}) = \textstyle\sum_{k=2}^{j-1} \tfrac{1}{(j-1-k)!} (-\sigma(\omega))^{j-1-k} \Delta_{k+1}(\tilde{\omega}) = \\ &\textstyle\sum_{k=3}^{j} \tfrac{1}{(j-k)!} (-\sigma(\omega))^{j-k} \Delta_k(\tilde{\omega})\end{aligned}$$

and the thesis is proved (end of proof for lemma 5). ■

By Lemma 5 it follows that, for the generating set, it holds $\Delta_C = \{G(\omega)\Delta^f(\tilde{\omega}), \forall \omega \in \Omega, \tilde{\omega} \in \mathcal{L}\}$. Observe that Δ_C is not yet a finite basis (because Ω is an infinite free group). However a finite basis for Δ_C is provided by a deeper analysis as follows. Recall that $W \in \mathbb{Q}^{2\times 2}$. Then we write the components of the columns of WS_+, $w'_{i,j} = \frac{p_{i,j}}{q_{i,j}}$ with $p_{i,j}, q_{i,j}$ coprime integers, for $j = 1, 2$, and, by letting $d_{i,j}, p, q$ be integer numbers with p, q coprime, $\frac{p_{i,j}}{q_{i,j}} = d_{i,j}\frac{p}{q}$ $\forall i = 1, \dots, c'$ and $j = 1, 2$. Thus elements of $\mathcal{U}$ can be written as $w_i = \frac{p}{q}(\lambda_1 d_{i,1}, \lambda_2 d_{i,2})$, $i = 1, \dots, c'$. Then, if $\omega = v_1 \cdots v_N$, for some $\nu_i \in \mathbb{Z}$, one can write $\sigma(\omega) = \sum_{i=1}^{N} v_{i,1} = \sum_{i=1}^{c'} \nu_i w_{i,1} = \lambda_1 \frac{p}{q} \sum_{i=1}^{c'} \nu_i d_{i,1}$. Define $\kappa(\omega) \in \mathbb{Z}$ as $\kappa(\omega) = \sum_{i=1}^{c'} \nu_i d_{i,1}$, such that $\sigma(\omega) = \lambda_1 \frac{p}{q} \kappa(\omega)$. Observe that $\kappa(\omega) = -\kappa(\omega^{-1})$.

Lemma 6: Choose $\hat{\omega}_0, \dots, \hat{\omega}_{n-3}$ such that $\hat{\omega}_i \in \Omega$ and $\kappa(\hat{\omega}_i) = i$ and define $B = \{G(\hat{\omega}_0)\Delta^f(\tilde{\omega}), \cdots, G(\hat{\omega}_{n-3})\Delta^f(\tilde{\omega}) : \tilde{\omega} \in \mathcal{L}\}$. Then B, a finite set, generates Δ_C by integer linear combinations.

Proof: Fix $\tilde{\omega}$. To prove the proposition it is sufficient to show that for $\omega \in \Omega$ with $\kappa(\omega) > n - 3$ or $\kappa(\omega) < 0$, a positive linear integer combination of $G(\hat{\omega}_0), \cdots, G(\hat{\omega}_{n-3})$ exists such that $\sum_{i=0}^{n-3} b_i G(\hat{\omega}_i)\Delta^f(\tilde{\omega}_i) = G(\omega)\Delta^f(\tilde{\omega})$. Notice that this is equivalent to showing that a linear combination over the integers exists such that

$$\sum_{i=0}^{n-3} a_i G(\hat{\omega}_i) = G(\omega). \tag{10}$$

since one can take $b_i = a_i$, $\tilde{\omega}_i = \tilde{\omega}$ if $a_i \geq 0$, otherwise $b_i = -a_i$ and $\tilde{\omega}_i = \tilde{\omega}^{-1}$.

Observe that $G(\hat{\omega}_i)$ is in the form

$$G(\hat{\omega}_i) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots \\ -\lambda_1 \frac{p}{q} i & 1 & 0 & 0 & \cdots & \cdots \\ \frac{1}{2!}\lambda_1^2 \frac{p^2}{q^2} i^2 & -\lambda_1 \frac{p}{q} i & 1 & 0 & \cdots & \cdots \\ -\frac{1}{3!}\lambda_1^3 \frac{p^3}{q^3} i^3 & \frac{1}{2!}\lambda_1^2 \frac{p^2}{q^2} i^2 & -\lambda_1 \frac{p}{q} i & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

The fact that such Toeplitz matrices are completely specified by their first column implies that finding the solution of (10) is reduced to solving for the first column, i.e., if $k(\omega) = \nu$, solving the system of $n - 2$ equations

$$\sum_{i=0}^{n-3} a_i i^k = \nu^k, \quad k = 0, \dots, n-3 \tag{11}$$

in a_i, $i = 0, \dots, n-3$. The unique solution of (11) is in $\mathbb{Z}$. Indeed (11) can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mu_0 & \mu_1 & \cdots & \mu_{n-3} \\ \mu_0^2 & \mu_1^2 & \cdots & \mu_{n-3}^2 \\ \vdots & \vdots & & \vdots \\ \mu_0^{n-3} & \mu_1^{n-3} & \cdots & \mu_{n-3}^{n-3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-3} \end{bmatrix} = \begin{bmatrix} 1 \\ \nu \\ \nu^2 \\ \nu^3 \\ \cdots \end{bmatrix} \tag{12}$$

where $\mu_i = i$. Observe that the Vandermonde determinant of the matrix in (12) is $\prod_{0 \le i < j \le n-3}(\mu_j - \mu_i)$. By the Cramer rule, solutions are given by

$$\begin{aligned} a_k &= \frac{\prod_{0 \le i < k}(\nu - \mu_i) \prod_{k < j \le n-3}(\mu_j - \nu) \prod_{\substack{0 \le i < j \le n-3 \\ i,j \ne k}}(\mu_j - \mu_i)}{\prod_{0 \le i < j \le n-3}(\mu_j - \mu_i)} \\ &= \frac{\prod_{0 \le i < k}(\nu - i) \prod_{k < j \le n-3}(j - \nu)}{\prod_{0 \le i < k}(k - i) \prod_{k < j \le n-3}(j - k)} \end{aligned}$$

i.e., up to sign, by binomial coefficients, which are integers (end of proof for lemma 6). ■

We have thus obtained a finite set B of generators for $\tilde{\Omega}$. We are now in a position to show the following:

Lemma 7: There exists λ^f such that each element in $\mathcal{V}$ can be written as $\mathsf{diag}\,(\lambda^f)v$, for some v with rational components.

Proof: Recall that $w_{i,j} = \lambda_j d_{i,j} \frac{p}{q}$ for all $i = 1, \dots, c'$ and $j = 1, 2$. Then for $\omega = v_1 v_2 \cdots v_N$, let $\sigma_i = \sigma_i(\omega) = \sum_{k>i}^{N} v_{k,1} = \lambda_1 \frac{p}{q} \sum_{k>i}^{N} d_{\ell(k),1}$ and, $\kappa_i = \kappa_i(\omega) = \sum_{k>i}^{N} d_{\ell(k),1}$. Then one can write $\sigma_i = \lambda_1 \frac{p}{q} \kappa_i$ and, for $j \geq 3$,

$$\begin{aligned}&\Delta_j(\omega) = \\ &\tfrac{1}{(j-1)!} \tfrac{\lambda_2}{\lambda_1} \lambda_1^{j-1} \left(\tfrac{p}{q}\right)^{j-1} \textstyle\sum_{i=1}^{N} \tfrac{d_{\ell(i),2}}{d_{\ell(i),1}} \left((d_{\ell(i),1} + \kappa_i)^{j-1} - (\kappa_i)^{j-1} \right) = \\ &\tfrac{1}{(j-1)!} \tfrac{\lambda_2}{\lambda_1} \lambda_1^{j-1} \left(\tfrac{p}{q}\right)^{j-1} p_j(\omega)\end{aligned}$$

where

$$p_j(\omega) = \sum_{i=1}^{N} \frac{d_{\ell(i),2}}{d_{\ell(i),1}} \left((d_{\ell(i),1} + \kappa_i)^{j-1} - (\kappa_i)^{j-1} \right).$$

In particular, for $j = 3$

$$\Delta_3(\omega) = \frac{1}{2}\lambda_2\lambda_1\left(\frac{p}{q}\right)^2 p_3(\omega).$$

Recalling that $\Delta^f_{j-2} = \Delta_j$, one has, for $j \geq 3$ and for a control words in B

$$\begin{array}{l}\Delta_j(\hat{\omega}_i\tilde{\omega}\hat{\omega}_i^{-1}) = \left(G(\hat{\omega}_i)\Delta^f(\tilde{\omega})\right)_{j-2} = \\ \sum_{r=3}^{j}\frac{1}{(j-r)!}\left(-\lambda_1\frac{p}{q}i\right)^{j-r}\Delta_r(\tilde{\omega}) = \\ \frac{1}{(j-1)!}\frac{\lambda_2}{\lambda_1}\lambda_1^{j-1}\left(\frac{p}{q}\right)^{j-1}\sum_{r=3}^{j}\binom{j-1}{r-1}(-i)^{j-r}p_r(\tilde{\omega})\end{array}$$

and denoting

$$\rho_{j,i} = \rho_{j,i}(\tilde{\omega}) = \sum_{r=3}^{j}\binom{j-1}{r-1}(-i)^{j-r}p_r(\tilde{\omega}), \tag{13}$$

one writes

$$\Delta_j(\hat{\omega}_i\tilde{\omega}\hat{\omega}_i^{-1}) = \frac{1}{(j-1)!}\frac{\lambda_2}{\lambda_1}\lambda_1^{j-1}\left(\frac{p}{q}\right)^{j-1}\rho_{j,i}.$$

where $\rho_{j,i}$ depends on $\tilde{\omega} \in \mathcal{L}_\lambda$ and is an integer number. Then for all $\tilde{\omega} \in \mathcal{L}$ and $i = 0, \ldots, n-3$ $G(\hat{\omega}_i)\Delta^f(\tilde{\omega}) = \mathsf{diag}\,(\lambda^f)v_{i,\tilde{\omega}}$ with $\lambda^f = (\frac{\lambda_2}{\lambda_1}\lambda_1^2, \frac{\lambda_2}{\lambda_1}\lambda_1^3, \ldots, \frac{\lambda_2}{\lambda_1}\lambda_1^{n-1})$, and
$v_{i,\tilde{\omega}} = \left(\frac{1}{(2)!}\left(\frac{p}{q}\right)^2\rho_{3,i}(\tilde{\omega}), \frac{1}{(3)!}\left(\frac{p}{q}\right)^2\rho_{4,i}(\tilde{\omega}), \ldots \frac{1}{(n-1)!}\left(\frac{p}{q}\right)^{n-1}\rho_{n,i}(\tilde{\omega})\right)$
(end of proof for lemma 7). ■

From the above lemma it immediately follows that the reachable set of $z^+ = z + v$ with $v \in \mathcal{V}$ is a discrete set in in $\mathbb{R}^{n-2}$. To finalize the proof of claim 1 by applying theorem 8, we provide a more detailed description of the structure of the reachable set. In particular we give m linearly independent generators of the lattice.

Lemma 8: Let $t_3(\lambda) = \frac{1}{2}\lambda_1\lambda_2\left(\frac{p}{q}\right)^2\bar{p}_3(\lambda)$ with $\bar{p}_3(\lambda) = G.C.D\{p_3(\tilde{\omega}),\ \tilde{\omega} \in \mathcal{L}\}$, be the minimum translations that can be obtained in the first variable on the fiber space, using control inputs from Ω. Then the lattice on the fiber is generated by the vectors

$$\left\{\bar{e}_3(\lambda) = \begin{bmatrix} 0 \\ 0 \\ t_3(\lambda) \\ \star \\ \star \\ \vdots \\ \star \end{bmatrix}, \ldots, \bar{e}_j(\lambda) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ t_j(\lambda) \\ \star \\ \vdots \\ \star \end{bmatrix}, \bar{e}_n(\lambda) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ t_n(\lambda) \end{bmatrix}\right\}$$

where $t_j(\lambda) = \frac{\lambda_2}{2}\lambda_1^{j-2}\left(\frac{p}{q}\right)^{j-1}\bar{p}_3(\lambda)$.

In order to prove lemma 8, we need first the following

Lemma 9: Using the conventions that $0^0 = \binom{0}{0} = 0! = 1$, it holds

$$\beta(\ell, s) = \sum_{j=0}^{\ell}(-1)^j\binom{\ell}{\ell-j}j^s = \begin{cases} 0 & \text{if } s < \ell \\ (-1)^\ell\ell! & \text{if } s = \ell \end{cases}$$

The proof of this lemma is given in the appendix, while for lemma 8 we have the following

Proof: The vector $\bar{e}_3(\lambda)$ can be generated by a positive integer combination of elements of $\mathcal{L}$. Indeed for all $i = 1, \ldots, n-3$, $\Delta_3(\hat{\omega}_i\tilde{\omega}\hat{\omega}_i^{-1}) = \Delta_3(\tilde{\omega})$

Next, for all $j \geq 4$ we want to find $n-2$ integers $\zeta_i^j,\ \ i = 0, \ldots, n-3$ (we denote $\zeta^j = (\zeta_0^j, \ldots, \zeta_{n-3}^j)$) and a word of type

$$\bar{\omega}(\zeta^j) = \underbrace{\hat{\omega}_0\tilde{\omega}\hat{\omega}_0^{-1}}_{\zeta_0^j\ times}\ldots\underbrace{\hat{\omega}_i\tilde{\omega}\hat{\omega}_i^{-1}}_{\zeta_i^j\ times}\ldots\underbrace{\hat{\omega}_{n-3}\tilde{\omega}\hat{\omega}_{n-3}^{-1}}_{\zeta_{n-3}^j\ times}$$

such that $\Delta(\bar{\omega}(\zeta^j))$ is a vector with zero in the first $j-3$ components. Observing that

$$\Delta(\bar{\omega}(\zeta^j)) = \sum_{i=0}^{j-3}\zeta_i^j\Delta(\hat{\omega}_i\tilde{\omega}\hat{\omega}_i^{-1}) = \sum_{i=0}^{j-3}\zeta_i^j\left(G(\hat{\omega}_i)\Delta^f(\tilde{\omega})\right),$$

the problem of finding $\bar{\omega}(\zeta^j)$ is equivalent to find $n-2$ integers ζ_i^j such that $G_j = \sum_{i=0}^{n-3}\zeta_i^j G(\hat{\omega}_i)$ is a lower $(n-2)\times(n-2)$ triangular matrix of rank $n-j+1$, i.e. to solve the system:

$$\sum_{i=0}^{n-3}\zeta_i^j i^k = 0 \quad k = 0, \ldots, j-4$$

in the integers ζ_i^j.

One solution is given by $\zeta^j = (\zeta_0, \ldots, \zeta_{j-3}, 0, \ldots, 0)$ with $\zeta_i^j = (-1)^i\binom{j-3}{j-3-i},\ \ i = 0, \ldots, j-3$ (observe that $\zeta_i^j,\ i = 0, \ldots, j-3$ are the binomial coefficients with alternate signs for the Newton binomial of degree $j-3$). Indeed, for all $k < j-3$,

$$\sum_{i=0}^{j-3}(-1)^i\binom{j-3}{j-3-i}i^k = \beta(j-3, k) = 0$$

by Lemma 9. Moreover, for $k = j-3$

$$\sum_{i=0}^{n-3}\zeta_i^j i^{j-3} = \beta(j-3, j-3) = (-1)^{j-3}(j-3)!.$$

Observe that, by the structure of the matrices $G(\hat{\omega}_i)$, the components on each diagonal of G_j are all equal. In particular the first non zero diagonal of G_j is the one corresponding to the $(j-2)$–th row with value $(-1)^{j-3}\frac{1}{(j-3)!}\lambda_1^{j-3}\left(\frac{p}{q}\right)^{j-3}\beta(j-3, j-3) = \lambda_1^{j-3}\left(\frac{p}{q}\right)^{j-3}$. Hence the first non zero component of $G_j\Delta^f(\tilde{\omega})$ is

$$t_j = \lambda_1^{j-3}\left(\frac{p}{q}\right)^{j-3}\Delta_3(\tilde{\omega}) = \frac{1}{2}\lambda_2\lambda_1^{j-2}\left(\frac{p}{q}\right)^{j-1}p_3(\tilde{\omega})$$

and, passing to the $G.C.D.$ over $\tilde{\omega} \in \mathcal{L}$ one obtains the expression for $t_j(\lambda)$ hence for $\bar{e}_j(\lambda)$.

To complete the proof it remains to show that t_j is the minimum that can be achieved so that $\bar{e}_j(\lambda) = G_j\Delta^f(\tilde{\omega})$.

First of all we will prove that for all $k \leq j-3$, and for all $\nu > j-3$, ν^k can be written as an integer linear combination of i^k, for $i = 1, \ldots, j-3$. In other words there exists integers $a_0, \ldots, a_{j-3}$ such that for all $k = 0, \ldots, j-3$,

$$\nu^k = \sum_{i=0}^{j-3} a_i(\nu) i^k,$$

for $\nu = j-2, \ldots, n-3$. We have a unique solution, indeed rewriting the equation in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & j-3 \\ 0 & 1^2 & 2^2 & \cdots & (j-3)^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1^{j-3} & 2^{j-3} & \cdots & (j-3)^{j-3} \end{bmatrix} \begin{bmatrix} a_0(\nu) \\ a_1(\nu) \\ a_2(\nu) \\ \vdots \\ a_{j-3}(\nu) \end{bmatrix} = \begin{bmatrix} 1 \\ \nu \\ \nu^2 \\ \vdots \\ \nu^{j-3} \end{bmatrix}$$

we have that this is exactly equation (11) where we have replaced n by j. Then

$$\sum_{i=0}^{n-3} b_i i^k = 0 \quad k = 0, \ldots, j-4$$

can be rewritten as

$$\sum_{i=0}^{j-3} \left(b_i + \sum_{\nu=j-2}^{n-3} a_i(\nu) b_\nu \right) i^k = 0 \quad k = 0, \ldots, j-4$$

i.e. a system with a one dimensional space of solutions. Hence for all $i = 0, \ldots, j-3$, $b_i + \left(\sum_{\nu=j-2}^{n-3} a_i(\nu) b_\nu \right) = \mu \zeta_i^j$ where μ must be an integer because $|\zeta_1^j| = |\zeta_{j-3}^j| = 1$ and all numbers in the righthandside are integers. Then any other solution gives rise to the translation

$$\sum_{i=0}^{n-3} b_i i^{j-3} = \sum_{i=0}^{j-3} \mu \zeta_i^j i^{j-3} = \mu t_j$$

i.e. the minimum is for $\mu = 1$ which finalizes the proof (end of proof for lemma 8). ■

We can now apply theorem 8 with $M = 1$, to conclude that the reachable set is a lattice. The proof of claim 1 is now completed. ■

We are finally ready to prove cases 2) and 3) of theorem 9. While the thesis has already been proved for the base system, we restate here the claim on the fiber system for convenience.

Claim 2: The reachable set of the fiber subsystem of (8) under a regular quantized control set $\mathcal{U} = \cup_{i=1}^M \mathcal{W}_i$ is dense if $M \geq 2$.

Proof: Recall that for all $\lambda \in \{\lambda_1, \ldots, \lambda_M\}$

$$\bar{e}_j(\lambda) = \sum_{i=0}^{j-3} \zeta_i^j G(\hat{\omega}_i) \Delta^f(\tilde{\omega}) = \sum_{i=0}^{j-3} \zeta_i^j \Delta(\hat{\omega}_i \tilde{\omega} \hat{\omega}_i^{-1}),$$

then the expression for the r–th component of $\bar{e}_j(\lambda)$ is:

$$\frac{1}{(r-1)!} \frac{\lambda_2}{\lambda_1} \lambda_1^{r-1} \left(\frac{p}{q} \right)^{r-1} \rho_r$$

where

$$\rho_r = \sum_{i=0}^{n-3} \zeta_i^j \rho_{r,i}.$$

is an integer number.

Then, for each $\lambda \in \{\lambda_1, \ldots, \lambda_M\}$, there exist integers $\nu_j(\lambda), a_3(\lambda), \ldots, a_n(\lambda)$ such that

$$\sum_{i=3}^{n} a_i(\lambda) \bar{e}_j(\lambda) = \nu_j(\lambda) t_j(\lambda) e_j$$

where e_j is the j–th element of the canonical base for $\mathbb{R}^n$ and $t_j(\lambda)$ is defined in lemma 8.

For each $i = 1, \ldots, M$ we denote Ω_{λ_i} as the word of input symbols for $\mathcal{W}_i = (\lambda_i, W_i, S_i)$ and in similar way we denote $\mathcal{L}_{\lambda_i}$, C_{λ_i}, $\Delta_{C_{\lambda_i}}$, B_{λ_i}, ...

Observe that, for all $j = 3, \ldots, n$, $\nu_j(\lambda_i) t_j(\lambda_i) e_j$ belongs to the lattice generated by B_{λ_i}. Moreover we can write:

$$\nu_j(\lambda_i) t_j(\lambda_i) e_j = \mathsf{diag}\,(\lambda_i^f) W_i^f s_{i,j}^f$$

where

$$\lambda_i^f = (\lambda_{i,2}\lambda_{i,1}, \lambda_{i,2}\lambda_{i,1}^2, \ldots, \lambda_{i,2}\lambda_{i,1}^{n-2}),$$

$$W_i^f = \frac{1}{2}\, \mathsf{diag} \left(\left(\frac{p}{q}\right)^2, \left(\frac{p}{q}\right)^3, \ldots, \left(\frac{p}{q}\right)^{n-1} \right)$$

$$s_{i,j}^f = \nu_j(\lambda_i) \bar{p}_3(\lambda_i) e_j,$$

and

$$S_i^f = \{s_{i,j}^f,\ j = 3, \ldots n\}.$$

Observe that $\cup_{i=1}^M \mathcal{W}_i^f \subset \mathcal{V}$ with $\mathcal{W}_i^f$ corresponding to $(\lambda_i^f, W_i^f, S_i^f)$. Moreover, any element of type $\Delta(\omega\tilde{\omega}\omega^{-1})$ with $\omega \in \Omega_{\lambda_i}$ and $\tilde{\omega} \in \mathcal{L}_{\lambda_k}$, $i \neq k$, is an element of $\mathcal{V}$ not belonging to $\mathcal{W}_i^f$ for any $i = 1, \ldots, M$.

We want to apply theorem 8.2 to the set $\cup_{i=1}^M \mathcal{W}_i^f \subset \mathcal{V}$ which is regular: it is symmetric and $\{\nu_j(\lambda_i) t_j(\lambda_i) e_j, j = 3, \ldots n\}$ are linearly independent vectors in $\mathcal{W}_i$.

If $M \geq 2$ and condition **C.2** holds but not **C.1** then the reachable set of the fiber system is dense. Indeed let i, k such that $\frac{\lambda_{i,2}}{\lambda_{k,2}} \notin \mathbb{Q}$ then

$$\frac{\lambda_{i,j}^f}{\lambda_{k,j}^f} = \frac{\lambda_{i,2}}{\lambda_{k,2}} \left(\frac{\lambda_{i,1}}{\lambda_{k,1}} \right)^{j-2} \notin \mathbb{Q}$$

for all $j = 3, \ldots, n$. If otherwise, only condition **C.1** holds, we have to analyse the ratios $\left(\frac{\lambda_{i,1}}{\lambda_{k,1}}\right)^{j-2}$. Consider the following condition:

$$[\mathbf{C.1.s}]\ \exists\, i, k \in \{1, \ldots, a\} \text{ such that } \left(\frac{\lambda_{i,1}}{\lambda_{k,1}}\right)^s \notin \mathbb{Q}.$$

Let

$$\mathcal{S} = \{s \geq 1, \ \text{ s.t. condition } \ \mathbf{C.1.s} \text{ holds}\}$$

then the reachable set on the fiber is dense at least in the subspace of the fiber generated by $\{e_j,\ j-2 \in \mathcal{S}\}$ of dimension $|\mathcal{S}| = (n-2) - \left[\frac{n-2}{\bar{s}}\right]$ where $\bar{s}$ is the minimum of the complement of $\mathcal{S}$ in the set $\{1, \ldots, n-2\}$. Observe that $\bar{s} > 1$ by hypothesis.

If both conditions **C.1** and **C.2** of theorem 8 hold we have to analyse the ratios $\frac{\lambda_{i,2}}{\lambda_{k,2}} \left(\frac{\lambda_{i,1}}{\lambda_{k,1}}\right)^{j-2}$. Consider the following condition:

[C.1'.s] $\exists\, i,k \in \{1,\ldots,a\}$ such that $\frac{\lambda_{i,2}}{\lambda_{k,2}}\left(\frac{\lambda_{i,1}}{\lambda_{k,1}}\right)^s \notin \mathbb{Q}$.

and let

$$\mathcal{R} = \{s \geq 1, \ \text{ s.t. condition } \ \mathbf{C.1'.s} \text{ holds}\}$$

Then the reachable set on the fiber is dense in at least a subspace generated by $\{e_j,\ j-2 \in \mathcal{R}\}$ of dimension $|\mathcal{R}| = (n-2) - \left[\frac{n-2}{\bar{r}}\right]$ where $\bar{r}$ is the minimum of the complement of $\mathcal{R}$ in the set $\{1,\ldots,n-2\}$.

To complete the proof we need to analyse the following cases:
1. Only condition **C.1** holds and there exists j s.t. $j-2 \notin \mathcal{S}$
2. Both conditions **C.1** and **C.2** hold and there exists j s.t. $j-2 \notin \mathcal{R}$
We consider $\Delta_j(\hat{\omega}_\mu\tilde{\omega}\hat{\omega}_\mu^{-1})$ with $\hat{\omega}_\mu \in \Omega_{\lambda_i}$ and $\tilde{\omega} \in \tilde{\Omega}_{\lambda_k}$, $p_3(\tilde{\omega}) \neq 0$, and compare with the set

$$\{\nu_j(\lambda_i)t_j(\lambda_i), \nu_j(\lambda_k)t_j(\lambda_k),\ j=3,\ldots n\}.$$

$$\begin{aligned}\Delta_j(\hat{\omega}_\mu\tilde{\omega}\hat{\omega}_\mu^{-1}) &= \left(G(\hat{\omega}_\mu)\Delta^f(\tilde{\omega})\right)_{j-2} = \\ &\textstyle\sum_{r=3}^{j}\frac{1}{(j-r)!}\left(-\lambda_{i,1}\frac{p}{q}\mu\right)^{j-r}\Delta_r(\tilde{\omega}) = \\ &\textstyle\sum_{r=3}^{j}\frac{1}{(j-r)!}\left(-\lambda_{i,1}\frac{p}{q}\mu\right)^{j-r}\frac{1}{(r-1)!}\frac{\lambda_{k,2}}{\lambda_{k,1}}\lambda_{k,1}^{r-1}\left(\frac{p}{q}\right)^{r-1}p_r(\tilde{\omega}) = \\ &\textstyle\frac{1}{(j-1)!}\left(\frac{p}{q}\right)^{j-1}\lambda_{k,2}\lambda_{k,1}^{j-2}\sum_{r=3}^{j}\binom{j-1}{r-1}\left(-\frac{\lambda_{i,1}}{\lambda_{k,1}}\mu\right)^{j-r}p_r(\tilde{\omega}) = \\ &\textstyle\frac{1}{(j-1)!}\left(\frac{p}{q}\right)^{j-1}\lambda_{k,2}\lambda_{k,1}^{j-2}\rho_{j,\alpha\mu}\end{aligned}$$

with $\alpha = \frac{\lambda_{i,1}}{\lambda_{k,1}}$. Recalling equation (13), we prove that $\rho_{j,\alpha\mu} \notin \mathbb{Q}$, $\forall \mu \neq 0$ and $\forall j$ s.t. $j-2 \notin \mathcal{S}$. Notice that α is a root of a polynomial $p(x) = x^{\bar{s}} - q$ where $q \in \mathbb{Q}$. Then $p(x)$ is the minimal polynomial of α, indeed if the minimal polynomial would have degree $s < \bar{s}$ then its term of degree 0 would be a rational number and a product of roots of $p(x)$ thus we would get $\alpha^s \in \mathbb{Q}$ contradicting the minimality of $\bar{s}$. Assume, by contradiction, that there exists $\xi \in \mathbb{Q}$ such that $\rho_{j,\alpha\mu} - \xi = 0$. If $j-2 = \bar{s}$ then there exists a polynomial with rational coefficients of degree $\bar{s}-1$, with α as a root. But the degree of the minimal polynomial of α is $\bar{s}$ which is a contradiction. If otherwise $j-2$ is a multiple of $\bar{s}$ we can write $\rho_{j,\alpha\mu} - \xi$ as a polynomial with rational coefficients of degree $\bar{s}-1$ in α by substituting $\alpha^s = (\alpha^{\bar{s}})^n\alpha^{s'}$ where $s = n\bar{s}+s'$, $s' < \bar{s}$ and $(\alpha^{\bar{s}})^n$ is a rational number. Then by the arguments used before we obtain the same conclusion.

We now prove that $\{1, \rho_{j,\alpha\mu},\ \ j-2 \notin \mathcal{S}\}$ are linearly independent over $\mathbb{Z}$. Indeed for every $b_j \in \mathbb{Z}$, $b_0 + \sum_{j-2\notin\mathcal{S}} b_j\rho_{j,\alpha\mu}$ is a polynomial in α. By arguments used above, it is easy to check that it can be zero if and only if $b_j = 0$ for every j.

Analogously $\rho_{j,\alpha\mu} \notin \mathbb{Q}$, $\forall \mu \neq 0$ and $\forall j$ s.t. $j-2 \in \mathcal{R}$. Indeed if $j-2$ is a multiple of $\bar{r}$ and $j-2 < \bar{s}$ then $\rho_{j,\alpha\mu} - \xi$ would be a polynomial with rational coefficients of degree $\bar{r}-1 < \bar{s}$, with $\bar{s}$ the degree of the minimal polynomial of α. If otherwise $j-2 > \bar{s}$ then by substituting the rational value of $\alpha^{\bar{s}}$ we would find a polynomial with rational coefficients of degree strictly less than that of the minimal polynomial of α.

As before $\{1, \rho_{j,\alpha\mu}:\ j-2 \notin \mathcal{R}\}$ are linearly independent over $\mathbb{Z}$.

By choosing $\ell(j) = i$ or k, we can apply theorem 6, case ii), to

$$v_j = \nu_j(\lambda_{\ell(j)})t_j(\lambda_{\ell(j)})e_j, j-2 \notin \mathcal{S}[\mathcal{R}]$$

and

$$v_{|\mathcal{S}|}[v_{|\mathcal{R}|}] = \Delta(\hat{\omega}_\mu\tilde{\omega}\hat{\omega}_\mu^{-1}).$$

This concludes the proof of claim 2 of theorem 9. ∎

Corollary 1: Consider the system (8) and assume that $\mathcal{U} \subset \mathbb{Q}^2$ is a regular sufficiently rich control set. Given a vector λ with irrational components, for every $\epsilon > 0$ there exists $\delta > 0$ such that if there exists $u \in \mathcal{U}$ with $\|u-\lambda\| < \delta$, then the system is ϵ-approachable.

Proof: Since the components of λ are irrational, given $\epsilon > 0$ there exist $m, n \in \mathbb{Z}$ such that $|m + nt_j(\lambda)| < \frac{\epsilon}{2}$. If $\|u-\lambda\| < \delta$ then $|m + nt_j(u)| < C\delta + \frac{\epsilon}{2}$ and we conclude taking δ sufficiently small. ∎

V. Conclusions

In this paper, we have considered reachability problems in quantized control systems. We have shown that the reachable set may be dense or discrete depending on the quantized set of inputs, and have provided some results in the analysis and synthesis problems. We have also provided a definition and some characterization of nonholonomic phenomena occurring in nonlinear quantized control systems. Many open problems remain in this field, that is in our opinion among the most important and challenging for applications of embedded control systems and in several other applications. Although some problems have been shown to be hard, we believe that a reasonably complete and useful system theory of quantized control system could be built by merging modern discrete mathematics techniques with classical tools of system theory.

VI. Appendix

Proof of Lemma 2 We show the lemma, by induction on the length of ω. If ω is a word of length 1 then the forms of $A_j(\omega,x)$ and of $\Delta_j(\omega)$ follow trivially by equation (8). Let $\omega' = \omega v_{N+1}$ with ω a word of length N, $\sigma' = \sigma + v_{N+1,1}$, $\sigma_i' = \sigma_i + v_{N+1,1}$, $i = 1,\ldots,N$, and suppose that $x(N) = \mathcal{A}(\omega,x) = x + A(\omega,x) + \Delta(\omega)$. Then

$$\begin{aligned}x_j(N+1) &= x_j(N) + A_j(v_{N+1}, x(N)) + \Delta_j(v_{N+1}) = \\ &x_j(N) + \textstyle\sum_{r=2}^{j-1}\frac{1}{(j-r)!}x_r(N)v_{N+1,1}^{j-r} + \frac{1}{(j-1)!}v_{N+1,2}v_{N+1,1}^{j-2}\end{aligned}$$

and, substituting the expressions for $x_r(N)$, $r = 2,\ldots,j$ into the last equation we have

$$\begin{aligned}x_j(N+1) = &\Big(x_j + \textstyle\sum_{r=2}^{j-1}\frac{1}{(j-r)!}x_r\sigma^{j-r} + \\ &\textstyle\frac{1}{(j-1)!}\sum_{i=1}^{N}\frac{v_{i,2}}{v_{i,1}}\left((v_{i,1}+\sigma_i)^{j-1} - \sigma_i^{j-1}\right)\Big) + \\ &\textstyle\sum_{r=2}^{j-1}\frac{1}{(j-r)!}\Big(x_r + \sum_{s=2}^{r-1}\frac{1}{(r-s)!}x_s\sigma^{r-s} + \\ &\textstyle\frac{1}{(r-1)!}\sum_{i=1}^{N}\frac{v_{i,2}}{v_{i,1}}\left((v_{i,1}+\sigma_i)^{r-1} - \sigma_i^{r-1}\right)\Big)v_{N+1,1}^{j-r} + \\ &\textstyle\frac{1}{(j-1)!}v_{N+1,2}v_{N+1,1}^{j-2}.\end{aligned}$$

Collecting together the terms that depend on x and those that do not, we obtain $x_j(N+1) = x_j + A_j'(\omega',x) + \Delta_j'(\omega')$

with

$$A'_j(\omega',x) = \sum_{r=2}^{j-1} \frac{1}{(j-r)!} x_r \sigma^{j-r} + \sum_{r=2}^{j-1} \frac{1}{(j-r)!} \left(x_r + \sum_{s=2}^{r-1} \frac{1}{(r-s)!} x_s \sigma^{r-s} \right) v_{N+1,1}^{j-r}$$

and

$$\Delta'_j(\omega') = \frac{1}{(j-1)!} \sum_{i=1}^{N} \frac{v_{i,2}}{v_{i,1}} \left((v_{i,1}+\sigma_i)^{j-1} - \sigma_i^{j-1} \right) + \sum_{r=2}^{j-1} \frac{v_{N+1,1}^{j-r}}{(j-r)!(r-1)!} \left(\sum_{i=1}^{N} \frac{v_{i,2}}{v_{i,1}} \left((v_{i,1}+\sigma_i)^{r-1} - \sigma_i^{r-1} \right) \right) + \frac{1}{(j-1)!} v_{N+1,2} v_{N+1,1}^{j-2}.$$

We show first that $A'(\omega',x) = A(\omega',x)$ and afterwords that $\Delta'(\omega') = \Delta(\omega')$. In a more compact way we can write

$$A'_j(\omega',x) = \sum_{r=2}^{j-1} \frac{1}{(j-r)!} x_r \left(\sigma^{j-r} + v_{N+1,1}^{j-r} \right) + \sum_{r=2}^{j-1} \sum_{s=2}^{r-1} \frac{1}{(j-s)!} \binom{j-s}{j-r} x_s \sigma^{r-s} v_{N+1,1}^{j-r}$$

Observing that

$$\sum_{r=2}^{j-1} \sum_{s=2}^{r-1} \frac{1}{(j-s)!} \binom{j-s}{j-r} x_s \sigma^{r-s} v_{N+1,1}^{j-r} = \sum_{s=2}^{j-2} \frac{1}{(j-s)!} x_s \sum_{r-s=1}^{(j-s)-1} \binom{j-s}{r-s} \sigma^{r-s} v_{N+1,1}^{(j-s)-(r-s)} = \sum_{s=2}^{j-2} \frac{1}{(j-s)!} x_s \left((\sigma + v_{N+1,1})^{(j-s)} - \sigma^{(j-s)} - v_{N+1,1}^{(j-s)} \right),$$

and noticing that in the last line we can replace the sum with that up to $j-1$ thus it follows

$$A'_j(\omega',x) = \sum_{r=2}^{j-1} \frac{1}{(j-r)!} x_r \left[\left(\sigma^{j-r} + v_{N+1,1}^{j-r} \right) + \left((\sigma + v_{N+1,1})^{(j-r)} - \sigma^{(j-r)} - v_{N+1,1}^{(j-r)} \right) \right] = \sum_{r=2}^{j-1} \frac{1}{(j-r)!} x_r \left(\sigma + v_{N+1,1} \right)^{(j-r)} = \sum_{r=2}^{j-1} \frac{1}{(j-r)!} x_r (\sigma')^{(j-r)} = A_j(\omega',x).$$

Next observe that, since by definition $\sigma'_{N+1} = 0$, $\Delta'_j(\omega',x)$ can be written as:

$$\Delta'_j(\omega) = \frac{1}{(j-1)!} \left[\sum_{i=1}^{N} \frac{v_{i,2}}{v_{i,1}} \left((v_{i,1}+\sigma_i)^{j-1} - \sigma_i^{j-1} \right) + \sum_{r=2}^{j-1} \binom{j-1}{j-r} \left(\sum_{i=1}^{N} \frac{v_{i,2}}{v_{i,1}} \left((v_{i,1}+\sigma_i)^{r-1} - \sigma_i^{r-1} \right) \right) v_{N+1,1}^{j-r} + \frac{v_{N+1,2}}{v_{N+1,1}} \left((v_{N+1,1} + \sigma'_{N+1})^{j-1} - (\sigma'_{N+1})^{j-1} \right) \right].$$

Consider the second sum appearing in $\Delta'_j(\omega)$; by reversing the order of the sums we can write:

$$\sum_{r=2}^{j-1} \binom{j-1}{j-r} \left(\sum_{i=1}^{N} \frac{v_{i,2}}{v_{i,1}} \left((v_{i,1}+\sigma_i)^{r-1} - \sigma_i^{r-1} \right) \right) v_{N+1,1}^{j-r} = \sum_{i=1}^{N} \frac{v_{i,2}}{v_{i,1}} \sum_{r=2}^{j-1} \binom{j-1}{j-r} \left((v_{i,1}+\sigma_i)^{r-1} - \sigma_i^{r-1} \right) v_{N+1,1}^{j-r},$$

where

$$\sum_{r=2}^{j-1} \binom{j-1}{j-r} \left((v_{i,1}+\sigma_i)^{r-1} - \sigma_i^{r-1} \right) v_{N+1,1}^{j-r} = \sum_{r-1=1}^{j-2} \binom{j-1}{(j-1)-(r-1)} v_{N+1,1}^{(j-1)-(r-1)} (v_{i,1}+\sigma_i)^{r-1} - \sum_{r-1=1}^{j-2} \binom{j-1}{(j-1)-(r-1)} v_{N+1,1}^{(j-1)-(r-1)} \sigma_i^{r-1} = \left[(v_{N+1,1} + (v_{i,1}+\sigma_i))^{j-1} - v_{N+1,1}^{j-1} - (v_{i,1}+\sigma_i)^{j-1} \right] - \left[(v_{N+1,1}+\sigma_i)^{j-1} - v_{N+1,1}^{j-1} - \sigma_i^{j-1} \right] = (v_{i,1}+\sigma'_i)^{j-1} - (v_{i,1}+\sigma_i)^{j-1} - (\sigma'_i)^{j-1} + \sigma_i^{j-1}.$$

And, finally,

$$\Delta'_j(\omega) = \frac{1}{(j-1)!} \sum_{i=1}^{N} \frac{v_{i,2}}{v_{i,1}} \left((v_{i,1}+\sigma'_i)^{j-1} - (\sigma'_i)^{j-1} \right) + \frac{1}{(j-1)!} \frac{v_{N+1,2}}{v_{N+1,1}} \left[\left(v_{N+1,1} + \sigma'_{N+1} \right)^{j-1} - \left(\sigma'_{N+1} \right)^{j-1} \right] = \frac{1}{(j-1)!} \sum_{i=1}^{N+1} \frac{v_{i,2}}{v_{i,1}} \left((v_{i,1}+\sigma'_i)^{j-1} - (\sigma'_i)^{j-1} \right) = \Delta_j(\omega)$$

which completes the proof. □

Proof of Lemma 4 Let $\bar{\omega} \overset{def}{=} \omega^{-1}$, and in particular $\bar{w} = -w$.

Step 1. First of all we shall prove that if $\tilde{\omega}$ is comprised of elements of C then $\forall \omega \in \Omega$ $\omega\tilde{\omega}\bar{\omega}$ itself is comprised of elements of C. By definition for $\tilde{\omega} \in \mathcal{L}$ and $\forall \omega_1 \in \Omega$, $\tilde{\omega}_1 = \omega_1 \tilde{\omega} \bar{\omega}_1 \in C$. Then, clearly, $\forall \omega_2 \in \Omega$,

$$\omega_2 \tilde{\omega}_1 \bar{\omega}_2 = (\omega_2\omega_1) \tilde{\omega} (\omega_2\omega_1)^{-1} \in C.$$

Further, if $\tilde{\omega}_1, \ldots, \tilde{\omega}_N$ are elements of C and $\omega \in \Omega$ then

$$\omega\tilde{\omega}_1 \cdots \tilde{\omega}_N \bar{\omega} = (\omega\tilde{\omega}_1\bar{\omega}) \cdots (\omega\tilde{\omega}_i\bar{\omega}) \cdots (\omega\tilde{\omega}_N\bar{\omega})$$

is comprised of elements in C.

Step 2. Next we will show that if $\omega_1, \omega_2 \in \Omega$ then $\omega_1\omega_2\bar{\omega}_1\bar{\omega}_2$ belongs to the group generated by C. We shall see it by induction.

a) First we show that for any $v_1, v_2 \in \mathcal{U}$ $v_1 v_2 \bar{v}_1 \bar{v}_2$ belongs to the group generated by C. There exists $v_3 \in \mathcal{U}$ such that $pv_3 = mv_1 + nv_2$ with $p, m, n \in \mathbb{Z}$. Since $\mathcal{U}$ is symmetric we can, for simplicity, assume that $p, m, n \in \mathbb{N}$. Then $\omega v_1 v_2 \bar{v}_1 \bar{v}_2 \bar{\omega} = \omega' \omega''$ where

$$\omega = \underbrace{v_1 \cdots v_1}_{m-1 \ times}$$
$$\omega' = \underbrace{v_1 \cdots v_1}_{m \ times} \underbrace{v_2 \cdots v_2}_{n \ times} \underbrace{\bar{v}_3 \cdots \bar{v}_3}_{p \ times} \in \mathcal{L}$$
$$\omega'' = \underbrace{v_3 \cdots v_3}_{p \ times} \underbrace{\bar{v}_2 \cdots \bar{v}_2}_{n-1 \ times} \bar{v}_1 \bar{v}_2 \underbrace{\bar{v}_1 \cdots \bar{v}_1}_{m-1 \ times} \in \mathcal{L}.$$

b) Next step is to see that if $v_1 \in \mathcal{U}$ and $\omega_2 \in \Omega$ then property $(*)$

$$(*) \quad v_1 \omega_2 \bar{v}_1 \bar{\omega}_2 \text{ belongs to the group generated by } C.$$

holds true. The proof follows by induction on the length of ω_2. For length$(\omega_2) = 1$ property $(*)$ has been shown in a). Suppose that we have proved $(*)$ for all ω_2 with length strictly less than N. Suppose now that length of ω_2 is equal to N.

Let $\omega_2 = v_2 \omega'_2$ then

$$v_1 \omega_2 \bar{v}_1 \bar{\omega}_2 = (v_1 v_2 \bar{v}_1 \bar{v}_2) v_2 v_1 (\omega'_2 \bar{v}_1 \bar{\omega}'_2 v_1) \bar{v}_1 \bar{v}_2.$$

Observe that the elements in the parenthesis belong to the group generated by C by a) and by induction. We conclude applying **Step 1.**

c) Finally $\omega_1, \omega_2 \in \Omega$ then property $(**)$

$$(**) \quad \omega_1\omega_2\bar{\omega}_1\bar{\omega}_2 \text{ belongs to the group generated by } C.$$

holds true Again we shall prove it by induction on the length of ω_1. If length$(\omega_1) = 1$ recall the proof in **b)**. Suppose that we have proved $(**)$ for all ω_1 with length strictly less than N. Suppose now that length of ω_1 is equal to N. Let $\omega_1 = \omega_1' v_1$

$$\begin{aligned}&\omega_1\omega_2\bar{\omega}_1\bar{\omega}_2 = \\ &\omega_1'(v_1\omega_2)\bar{v}_1\bar{\omega}_1'\bar{\omega}_2 = \\ &\omega_1'(v_1\omega_2\bar{v}_1\bar{\omega}_2)\omega_2\bar{\omega}_1'\bar{\omega}_2 = \\ &\omega_1'(v_1\omega_2\bar{v}_1\bar{\omega}_2)(\omega_2\bar{\omega}_1'\bar{\omega}_2\omega_1')\bar{\omega}_1'\end{aligned}$$

The two terms in the parenthesis are elements of the group generated by C (by induction). Then the proof of **Step 2.** is completed.

Step 3. $\forall \omega \in \Omega$ and $\omega' \in \Omega$ with $\omega \equiv \omega'$ there exists some $\mathbf{g}$ belonging to the group generated by C such that $\omega = \mathbf{g}\omega'$. In other words $\omega = \omega'(\text{mod } C)$. By induction.

a. $\omega = v_1v_2$ then $\omega = (v_1v_2\bar{v}_1\bar{v}_2)v_2v_1$ with $(v_1v_2\bar{v}_1\bar{v}_2)$ an element of the group generated by C.

b. $\omega = v_1\mathbf{g}v_2$ with $\mathbf{g} = v_3v_4\bar{v}_3\bar{v}_4 \in C$ then $\omega = v_1v_2(\text{mod } C)$.

$$\begin{aligned}&v_1(v_3v_4\bar{v}_3\bar{v}_4)v_2 = \\ &(v_1v_3\bar{v}_1\bar{v}_3)v_3(v_1v_4\bar{v}_1\bar{v}_4)v_4 \\ &(v_1\bar{v}_3\bar{v}_1v_3)\bar{v}_3(v_1\bar{v}_4\bar{v}_1v_4)\bar{v}_4v_1v_2\end{aligned}$$

Let $[u, v]$ denote the commutator $uv\bar{u}\bar{v}$. For completing the proof we should prove that

$$[v_1, v_3]v_3[v_1, v_4]v_4[v_1, \bar{v}_3]\bar{v}_3[v_1, \bar{v}_4]\bar{v}_4 \in C$$

$$\begin{aligned}&[v_1, v_3]v_3[v_1, v_4]v_4[v_1, \bar{v}_3]\bar{v}_3[v_1, \bar{v}_4]\bar{v}_4 = \\ &[v_1, v_3]v_3[v_1, v_4]v_4[v_1, \bar{v}_3]\bar{v}_3\bar{v}_4 \\ &(v_4[v_1, \bar{v}_4]\bar{v}_4) = \\ &[v_1, v_3]v_3[v_1, v_4]v_4\bar{v}_3\bar{v}_4 \\ &(v_4v_3[v_1, \bar{v}_3]\bar{v}_3\bar{v}_4)(v_4[v_1, \bar{v}_4]\bar{v}_4) = \\ &[v_1, v_3]v_3v_4\bar{v}_3\bar{v}_4(v_4v_3\bar{v}_4[v_1, v_4]v_4\bar{v}_3\bar{v}_4) \\ &(v_4v_3[v_1, \bar{v}_3]\bar{v}_3\bar{v}_4)(v_4[v_1, \bar{v}_4]\bar{v}_4) = \\ &[v_1, v_3][v_3v_4](v_4v_3\bar{v}_4[v_1, v_4]v_4\bar{v}_3\bar{v}_4) \\ &(v_4v_3[v_1, \bar{v}_3]\bar{v}_3\bar{v}_4)(v_4[v_1, \bar{v}_4]\bar{v}_4)\end{aligned}$$

which is comprised of elements of C for what we have seen in **Step 1.**.

c. $\omega = v_1\mathbf{g}v_2$ with $\mathbf{g} = \omega[v_3v_4]\bar{\omega} \in \Omega$ then $\omega = v_1v_2(\text{mod } C)$. Suppose first that length$(\omega) = 1$ then

$$\begin{aligned}&v_1\omega[v_3v_4]\bar{\omega}v_2 = [v_1\omega]\omega v_1[v_3v_4]\bar{\omega}v_2 = \\ &[v_1\omega](\omega v_1[v_3v_4]\bar{v}_1\bar{\omega})\omega v_1\bar{\omega}v_2 = \\ &[v_1\omega](\omega v_1[v_3v_4]\bar{v}_1\bar{\omega})[\omega v_1]v_1\omega\bar{\omega}v_2 = \\ &[v_1\omega](\omega v_1[v_3v_4]\bar{v}_1\bar{\omega})[\omega v_1]v_1v_2\end{aligned}$$

Next suppose that for all $\omega = v_1\mathbf{g}v_2$ with $\mathbf{g} = \omega[v_3v_4]\bar{\omega} \in \Omega$ with lenght$(\omega) < K$, it holds $\omega = v_1v_2(\text{mod } C)$. We shall prove it also for lenght$(\omega) = K$. Let $\omega = v\omega'$ then

$$\begin{aligned}&v_1\omega[v_3v_4]\bar{\omega}v_2 = v_1v\omega'[v_3v_4]\bar{\omega}'\bar{v}v_2 = \\ &[v_1v]v(v_1\omega'[v_3v_4]\bar{\omega}')\bar{v}v_2\end{aligned}$$

By the inductive hypotheses (lenght$(\omega') < K$) one has:

$$\begin{aligned}&v_1\omega[v_3v_4]\bar{\omega}v_2 = [v_1v]v\mathbf{g}'v_1\bar{v}v_2 = \\ &[v_1v](v\mathbf{g}'\bar{v})vv_1\bar{v}v_2\end{aligned}$$

with $\mathbf{g}'$ comprised of elements of C. Finally

$$\begin{aligned}&v_1\omega[v_3v_4]\bar{\omega}v_2 = [v_1v](v\mathbf{g}'\bar{v})[vv_1]v_1v\bar{v}v_2 = \\ &[v_1v](v\mathbf{g}'\bar{v})[vv_1]v_1v_2\end{aligned}$$

and the proof is completed.

d. Let $\omega = v_1 \dots v_N$. Cleary by permuting the elements two by two any permutation of ω can be produced. Suppose the elements v_iv_{i+1} are permuted then, by letting $\omega_1 = v_1 \cdots v_{i-1}$ and $\omega_2 = v_{i+2} \cdots v_N$, one has

$$\omega_1v_iv_{i+1}\omega_2 = \omega_1[v_iv_{i+1}]v_{i+1}v_i\omega_2$$

If length$(\omega_1) = 1$ then by **c.** there exist some $\mathbf{g}$ comprised of elements of C either of type $[\cdot, \cdot]$ or of type $\omega[\cdot, \cdot]\bar{\omega}$ with $\omega \in \Omega$ such that

$$\omega_1[v_iv_{i+1}]v_{i+1}v_i\omega_2 = \mathbf{g}\omega_1v_{i+1}v_i\omega_2$$

Suppose that for length$(\omega_1) < K$ there exists some concatenation of elements of C, $\mathbf{g}$ either of type $[\cdot, \cdot]$ or of type $\omega[\cdot, \cdot]\bar{\omega}$ with $\omega \in \Omega$ such that

$$\omega_1[v_iv_{i+1}]v_{i+1}v_i\omega_2 = \mathbf{g}\omega_1v_{i+1}v_i\omega_2$$

Let now length$(\omega_1) = K$ and $\omega_1 = v_1\omega_1'$. Then

$$v_1\omega_1'[v_iv_{i+1}]v_{i+1}v_i\omega_2 = v_1\mathbf{g}\omega_1'v_{i+1}v_i\omega_2$$

Now $\mathbf{g}$ is comprised of elements of type $[\cdot, \cdot]$ and of type of type $\omega[\cdot, \cdot]\bar{\omega}$. We shall then use **b.** and **c.** to complete the proof.

Observe that if $\Sigma(\omega) = 0$ then $\omega = 0(\text{mod } C)$. In fact if $\Sigma(\omega) = 0$ then $\omega \equiv 0$.

Step 4. We shall now prove the proposition in the general case. Clearly if $\omega \in C$ then $\Sigma(\omega) = \Sigma(\tilde{\omega})$ for some $\tilde{\omega} \in \mathcal{L}$ and $\Sigma(\omega) = \pm(N_W)_j = N_W\alpha$ where $\alpha \in \mathbb{Z}^{c'-2}$ is a vector with all components zero except for the j–th which is ± 1. Therefore $\omega \in \tilde{\Omega}$.

Now, if $\omega \in \tilde{\Omega}$ then $\Sigma(\omega) = N_W\alpha$, for some $\alpha \in \mathbb{Z}^{c'-2}$. Let $v_{1,1}, \dots, v_{1,c'-2} \in \mathcal{L}$ be such that $\Sigma(v_{1,j}) = (N_W)_j$ and

$$\omega_1 = \underbrace{z_1 \dots z_1}_{|\alpha_1| times} \cdots \underbrace{z_{2-c'} \dots z_{2-c'}}_{|\alpha_{c'-2}| times}$$

where α_j is the j–th component of α and

$$z_j = \begin{cases} v_{1,j} & if \ \alpha_i > 0 \\ -v_{1,j} & if \ \alpha_i < 0. \end{cases}$$

Clearly $\Sigma(\omega_1) = N_W\alpha = \Sigma(\omega)$ and ω_1 concatenation of elements of C. By **Step 3.**, it is possible to permute (mod elements of C) the symbols ω so that $\omega \equiv \omega_0\omega_1$ with ω_0 a concatenation of elements of C of type $\omega'[\cdot, \cdot]\bar{\omega}'$. Then ω is comprised of elements of C. The proof is completed. $\square$

Proof of Lemma 9 First we show that for $s = 0$ it holds

$$\beta(\ell, 0) = \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{if } \ell \geq 1 \end{cases}$$

For $\ell = 0$ it follows trivially by the conventions.
For $\ell = 1$ then

$$\sum_{j=0}^{1} (-1)^j \binom{1}{j} = \binom{1}{1} - \binom{1}{0} = 0.$$

Suppose that $\ell > 1$ then

$$\sum_{j=0}^{\ell} (-1)^j \binom{\ell}{\ell - j} = \binom{\ell}{\ell} + \sum_{j=1}^{\ell-1} (-1)^j \binom{\ell}{\ell - j} + (-1)^\ell \binom{\ell}{0}.$$

Observe that, by the properties of binomial coefficients we have

$$\binom{\ell}{j} = \binom{\ell - 1}{j - 1} + \binom{\ell - 1}{j} \tag{14}$$

then

$$\begin{aligned}&\textstyle\sum_{j=0}^{\ell} (-1)^j \binom{\ell}{\ell-j} = \binom{\ell}{\ell} + \\ &\Big[-\left(\binom{\ell-1}{\ell-2} + \binom{\ell-1}{\ell-1} \right) + \left(\binom{\ell-1}{\ell-3} + \binom{\ell-1}{\ell-2} \right) - \ldots \\ &\qquad \ldots + (-1)^{l-1} \left(\binom{\ell-1}{0} + \binom{\ell-1}{1} \right) \Big] + \\ &(-1)^\ell \binom{\ell}{0}\end{aligned}$$

which reduces to

$$\begin{aligned}&\textstyle\sum_{j=0}^{\ell} (-1)^j \binom{\ell}{\ell-j} = \\ &\textstyle\binom{\ell}{\ell} + \left[-\binom{\ell-1}{\ell-1} + (-1)^{l-1} \binom{\ell-1}{0} \right] + (-1)^\ell \binom{\ell}{0} = 0.\end{aligned}$$

Suppose now that $s > 0$ then we show that the following holds true:

$$\beta(\ell, s) = \ell \beta(\ell, s - 1) - \ell \beta(\ell - 1, s - 1). \tag{15}$$

By collecting the first and the last terms of the sum which defines $\beta(\ell, s)$ we obtain

$$\begin{aligned}&\textstyle\beta(\ell, s) = 0^s + \sum_{j=1}^{\ell-1} (-1)^j \binom{\ell}{\ell-j} j^s + (-1)^\ell \ell^s = \\ &\textstyle(-1)^\ell \ell^s + \sum_{j=1}^{\ell-1} (-1)^j \ell \binom{\ell-1}{\ell-j} j^{s-1}\end{aligned}$$

and using equation (14) we have:

$$\begin{aligned}&\beta(\ell, s) = \\ &\textstyle(-1)^\ell \ell^s + \sum_{j=1}^{\ell-1} (-1)^j \ell \left(\binom{\ell}{\ell-j} - \binom{\ell-1}{(\ell-1)-j} \right) j^{s-1} = \\ &(-1)^\ell \ell^s + \\ &\textstyle\sum_{j=1}^{\ell-1} (-1)^j \ell \binom{\ell}{\ell-j} j^{s-1} - \sum_{j=1}^{\ell-1} (-1)^j \ell \binom{\ell-1}{(\ell-1)-j} j^{s-1}.\end{aligned}$$

By observing that the generic term of the two sums, for $j = 0$ either cancel each other (if $s = 1$) or they are both zero, while the generic term of the first sum, for $j = \ell$ is equal to $(-1)^\ell \ell^s$. Then one can write the following equation:

$$\begin{aligned}&\beta(\ell, s) = (-1)^\ell \ell^s + \\ &\textstyle\left(\sum_{j=0}^{\ell} (-1)^j \ell \binom{\ell}{\ell-j} j^{s-1} - (-1)^\ell \ell^s \right) - \\ &\textstyle\sum_{j=0}^{\ell-1} (-1)^j \ell \binom{\ell-1}{(\ell-1)-j} j^{s-1} = \\ &\ell \left(\beta(\ell, s - 1) - \beta(\ell - 1, s - 1) \right).\end{aligned}$$

Now we are ready to give the proof of the lemma by induction. For $0 < s < \ell$, by induction $\beta(\ell, s - 1) = 0$ because $\ell > s > s - 1$ and $\beta(\ell - 1, s - 1) = 0$ because $\ell - 1 > s - 1$. Then by equation (15) we obtain $\beta(\ell, s) = 0$.

For $0 < s = \ell$ it is an easy computation to check that $-\beta(1, 1) = \beta(0, 0) = 1$. Moreover for $s = \ell > 1$, we have $\beta(\ell, s - 1) = 0$ because $\ell > s - 1$ and, by the inductive hypothesis, $\beta(\ell - 1, s - 1) = (-1)^{\ell-1}(\ell - 1)!$. Then by equation (15) we obtain $\beta(\ell, s) = -\ell\beta(\ell-1, s-1) = (-1)^\ell \ell!$ which completes the proof.

□

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