Steering Driftless Nonholonomic Systems by Control Quanta*

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Abstract

We consider the problem of steering a class of nonholonomic systems, namely systems that are feedback equivalent to a strictly triangular form, which is considerably larger than other classes for which the steering problem has been given closed–form solutions in the literature. The proposed solution consists in the application of a finite concatenation of finite–support control actions chosen among a finite set, suitably selected in the input space, each resulting in a quantum change in the system state. The method results in a closed–form algorithm which is exact up to an arbitrary tolerance.

1 Introduction

Nonholonomic systems are intrinsically nonlinear systems – i.e., systems whose linearization destroys some structural property, and for which linear control techniques are inapplicable. Nonholonomy occurs in many systems, mainly but not exclusively mechanical, with vaste engineering relevance.

The practical importance of some nonholonomic systems, along with their theoretical importance as a class of "genuinely" nonlinear systems posing many challenging control problems, has stimulated the interest of researchers in the last decade. One very basic problem in noholonomic systems control, whose counterpart in linear systems control is easily solved, is steering the state between two given points (a constructive controlabillity problem).

To solve the steering (or "motion planning") problem for nonholonomic systems, researchers have proposed algorithms that can be subdivided in two classes, according to whether the algorithm is closed–form, or iterative. Closed–form algorithms are available for systems that possess some special structure: Murray and Sastry [12] used sinusoids to steer systems that are feedback–equivalent to chained–form; Rouchon *et al.* [14] use flat outputs for steering differentially flat systems; Monaco and Normand–Cyrot [11] apply nonlinear multirate control to systems that admit an exact sampled model (while maintaining controllability under sampling); Lafferriere and Sussman [7] use the Chien–Fliess–Sussmann equation to steer nilpotent systems. For more general systems, no closed–form algorithm is known at the state of the art. Iterative methods have been proposed e.g. by Sontag ([16]), Chitour and Sussmann ([4], [17]); Fernandes, Gurvits, and Li [6], Divelbiss and Wen, [5]. The Lafferriere–Sussman [7] algorithm can also be applied to non–nilpotent systems, guaranteeing in this case convergent successive approximations.

In this paper, we propose a closed–form method that allows steering driftless nonholonomic systems that are feedback–equivalent to the strictly triangular (ST) form

$$\begin{array}{rcl} \dot{\mathbf{x}}_1 & = & \mathbf{g}_1(\mathbf{x}_2,\ldots,\mathbf{x}_p)\mathbf{u} \\ \dot{\mathbf{x}}_2 & = & \mathbf{g}_2(\mathbf{x}_3,\ldots,\mathbf{x}_p)\mathbf{u} \\ & \vdots & \\ \dot{\mathbf{x}}_{p-1} & = & \mathbf{g}_{p-1}(\mathbf{x}_p)\mathbf{u} \\ \dot{\mathbf{x}}_p & = & \mathbf{u} \end{array} \tag{1}$$

with $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p] \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p} = \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^{n_p}$. The ST form represents a very large class of nonlinear driftless systems. The proposed method steers ST systems between the initial point and an arbitrarily small neighborhood of the final point by using a set of canonical paths, and might be regarded as a generalization of the Murray–Sastry sinusoid method. However, instead of using inputs from an n_p–parameter, "canonical" family tuned by explicit solution of the control ODE (which would not be possible for systems in more general form than chained), we use a finite combination of inputs chosen from a finite set, each providing a "quantum" motion of the system.

A motivating example for the quantized control approach is presented in section 2. Section 3 presents the steering algorithm for ST systems, and an example is discussed in section 4.

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2 Quantized control: a motivating example

In [1], the problem of steering a regular convex surface rolling on a plate was considered, with the purpose of building a dextrous hand with simplified hardware. This problem, first brought to the attention of the control community by R. Brockett, can be described by a 5–dimensional, 2–inputs driftless control system $\dot{\mathbf{x}} = \mathbf{g}_1(x)u_1 + \mathbf{g}_2(x)u_2$. This system is nonholonomic (equivalently, the distribution $\{\mathbf{g}_1, \mathbf{g}_2\}$ is not involutive). Furthermore, such system can not be put in chained form, is not differentially flat, and is not nilpotent. However, it has been shown in ([3]) that a regular feedback and a diffeomorphism exist, by which the system can be written in the ST form defined in (1).

The fact that an ODE in ST form can be integrated by quadratures, was exploited in [3] to devise a steering method which reduced to solving a system of algebraic nonlinear equations. However, the method was not completely satisfactory because integrals could not always be found in terms of simple functions. Moreover, solutions of the nonlinear algebraic equations remained to be found numerically.

A related problem to that of rolling a regular surface, which was imposed to the attention of the authors of [8] by practical applications to manipulation of genuine industrial parts, is rolling a polyhedral surface.

A polyhedron is said to "roll" on a plate when it rotates about one of the edges of the face being in contact with the plate, by the amount that exactly brings an adjacent face in contact. In this case, the configuration space of the system can be described by the 4-tuple $(x, y, \theta, F) \in \mathbf{M} = \mathbb{R}^2 \times S^1 \times \{F_1, ..., F_n\}$ where x, y are the coordinates of the projection on the plate of the center of the polyhedron, θ its orientation, and F identifies the face in touch with the plate. To each configuration, a finite set of possible control actions is associated, consisting in rolling to one of the adjacent faces.

It is interesting to observe that the study of rolling polyhedra suggests an alternative way of looking at the regular surface rolling problem, i.e. to consider the regular surface as the limit of a suitable sequence of convex polyhedra with an increasing number of faces [13]. The approximation introduces quantization of the input space, and as a consequence it is natural to consider a discrete time axis as well. However, the procedure would be substantially different from conventional sampling of the continuous time system. In practice, approximation of regular surfaces by polyhedra was not found to be a computationally viable solution; however, we will show how some ideas derived from the solution given to planning of simple polyhedral parts can be extrapolated to much more general problems.

The nonholonomic character of the regular rolling phenomenon can be observed also in the polyhedral case, provided that a more general definition of nonholonomy is accepted for systems where differential geometric definitions such as "involutivity of distributions" can not be introduced. One such definition is the following ([2]):

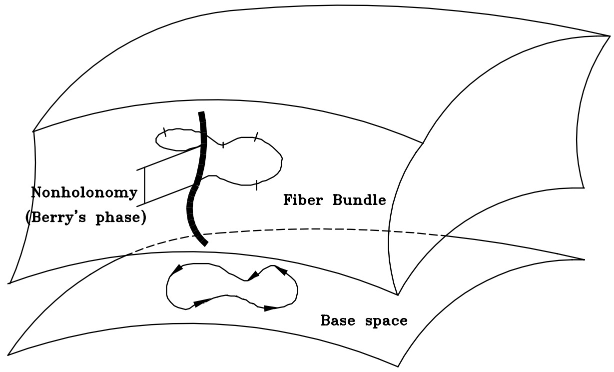

Figure 1: Illustrating the definition of nonholonomic systems

Definition 1 *Consider a system evolving in a configuration space $\mathcal{Q}$, a time set (continuous or discrete) $\mathcal{T}$, and a bundle of input sets $\mathcal{U}$, such that for each input set $\mathcal{U}(\mathbf{q}, \mathbf{t})$ defined at $\mathbf{q} \in \mathcal{Q}$, $\mathbf{t} \in \mathcal{T}$, it holds $\mathbf{u} : (\mathbf{q}, \mathbf{t}) \to \mathbf{q}'$, $\mathbf{q}' \in \mathcal{Q}, \forall \mathbf{u} \in \mathcal{U}(\mathbf{q}, \mathbf{t})$. If it is possible to decompose $\mathcal{Q}$ in a projection or* base *space $\mathcal{B} = \Pi(\mathcal{Q})$ and a* fiber bundle *$\mathcal{F}$, such that $\mathcal{B} \times \mathcal{F} = \mathcal{Q}$ and there exists a sequence of inputs in $\mathcal{U}$ starting at $\mathbf{q}_0$ and steering the system to $\mathbf{q}^\star = \mathbf{u}_n(\mathbf{q}_{n-1}, \mathbf{t}_{n-1}) \circ \cdots \circ \mathbf{u}_1(\mathbf{q}_0, \mathbf{t}_0)$, such that $\Pi(\mathbf{q}_0) = \Pi(\mathbf{q}^\star)$ but $\mathbf{q}_0 \neq \mathbf{q}^\star$, then the system is nonholonomic at $\mathbf{q}_0$.*

According to this definition, a system is nonholonomic if there exist controls that make some configurations go through closed cycles, while the rest of configurations undergo net changes per cycle (see figure 1).

The analysis of the reachability set for the rolling polyhedron system is more complex than for regular surfaces. The reachable set of a rolling polyhedron may exhibit either dense, or lattice, or mixed structure depending on values of some geometric parameters of the polyhedron (examples are an irregular tetrahedron, a cube, and a dodecahedron, respectively) (see [2]).

The solution to the problem of steering a system comprised of a rolling polyhedron on a plate has been solved in [8] using tools from the theory of groups. Observe that the manifold has a foliated structure, being comprised of n copies of $\mathbb{R}^2 \times S^1$, and a control input steers the system through different leaves. To all points on the same leaf the same set of admissible controls corresponds. Admissible controls are sequences of consecutively adjacent faces, starting with the face identifying the leaf; controls can be composed by concatenation whenever the final and starting faces of the sequences coincide.

Definition 1 specializes to the case of a rolling polyhedron by identifying the base space as the set of faces, $\mathcal{B} = \{F_1, \ldots, F_n\}$, and the fiber as $\mathcal{F} = \mathbb{R}^2 \times S^1$. The subsets $\mathcal{U}_1^i$ of controls beginning and ending with the i–th face are groups acting each on a single leaf of the configuration space, producing closed cycles on the base space, and, in general, net changes in the fiber variables (thus proving nonholonomy of the system). Moreover, this action is abelian on the orientation, i.e. the order of application of any two inputs does not change the final orientation of the polyhedron (this is not true for the position of the polyhedron on the plate).

The splitting process can now be applied again to the fiber $\mathbb{R}^2 \times S^1$ obtained at the first step, which is split in a new base (S^1) and a new fiber ($\mathbb{R}^2$). To this splitting there corresponds the subgroups $\mathcal{U}_2^i \subset \mathcal{U}_1^i$ of controls that produce no changes in orientation of the polyhedron. Subgroups $\mathcal{U}_2^i$ produce closed cycles on $\{F_1, \ldots, F_n\} \times S^1$, and their action on the fiber $\mathbb{R}^2$ is abelian (it is in fact a group of translations).

Each time a subgroup has an abelian action on some variables of the fiber, those fiber variables (which we will call henceforth "symmetric variables") can be easily steered. In fact, each control in the subgroup achieves a finite, known change in the fiber variables. Integer combinations of these quantum motions for symmetric variables can be obtained by concatenation of controls in the subgroup; notice that, being the system time–symmetric, quanta of negative sign are also available. For a polyhedron, the above defined subgroups of controls are generated by a finite number of elements, hence finitely many quanta of motion have to be considered. The set of configurations reachable by rolling being obtained as an integer combination of a finite number of elements, the closure of the reachable set itself may result to be the whole space or a discrete set, depending on relative rationality properties of the generating quanta.

In conclusion, the polyhedron steering problem is reduced to the following steps: i) reach the desired face F_i; ii) adjust the orientation (up to an approximation equal to the G.C.D. of orientation quanta) using controls in $\mathcal{U}_1^i$ by solving a one–dimensional Diophantine equation; iii) adjust the position (up to an approximation equal to the G.C.D. of translation quanta) using controls in $\mathcal{U}_2^i$ by solving a two–dimensional vectorial Diophantine equation.

At each step a different subset of components of the configuration space are brought to their final configuration. Then a subset of controls acting as closed cycles on the components already in the final configuration are used to steer another subset of components to their final configuration.

In the rest of this paper, we investigate under what conditions and how the above procedure can be proficuously extended to steering continuous nonholonomic systems.

3 Steering by control quanta

Consider a driftless, completely nonholonomic (i.e., controllable) system described by

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})\mathbf{u}(t), \ \mathbf{x} \in \mathbb{R}^n, \tag{2}$$

where controls $\mathbf{u}(\cdot)$ belong to a feasible set of functions with compact support in time $\mathcal{U}$ taking values in $\mathbb{R}^{n_p}$, and define the corresponding *end-point map* associated to an initial point $\mathbf{x}$ and to a control $\mathbf{u}$ as

$$\begin{array}{cccccc} \Sigma: & \mathbb{R}^n & \times & \mathcal{U} & \longrightarrow & \mathbb{R}^n \\ & (\mathbf{x} & , & \mathbf{u}) & \mapsto & \Sigma(\mathbf{x},\mathbf{u}) \end{array},$$

and introduce the notation $\Sigma_{\mathbf{u}}(\mathbf{x}) = \Sigma(\mathbf{x},\mathbf{u})$. Regarding $\mathcal{U}$ as the group of elements $\mathbf{u}_i : [0, T_i] \subset \mathbb{R}^+ \mapsto \mathbb{R}^{n_p}$ with the concatenation operation:

$$\mathbf{u}_2 \circ \mathbf{u}_1 = \left\{ \begin{array}{ll} \mathbf{u}_1(t) & \text{if } t \in [0, T_1] \\ \mathbf{u}_2(t - T_1) & \text{if } t \in [T_1, T_1 + T_2] \end{array} \right.,$$

and the inverse $\mathbf{u}_i^{-1}(t) = -\mathbf{u}_i(T_i - t)$, consider the group $\mathcal{A}(\mathbb{R}^n)$ of automorphisms of $\mathbb{R}^n$ with the operation

$$\begin{array}{ccccc} \mathbb{R}^n & \stackrel{\phi}{\to} & \mathbb{R}^n & \stackrel{\psi}{\to} & \mathbb{R}^n \\ \mathbf{z} & \mapsto & \phi(\mathbf{z}) & \mapsto & \psi(\phi(\mathbf{z})) \end{array}$$

Finally, let Φ be the homomorphism

$$\begin{array}{cccc} \Phi: & \mathcal{U} & \longrightarrow & \mathcal{A}(\mathbb{R}^n) \\ & \mathbf{u} & \mapsto & \Sigma_{\mathbf{u}} \end{array}$$

The homomorphism Φ is surjective because (2) is controllable by hypothesis, but not injective. However, the map

$$\bar{\Phi}: \quad \bar{\mathcal{U}} = \mathcal{U} / \ker \Phi \quad \longrightarrow \quad \mathcal{A}(\mathbb{R}^n)$$

is an isomorphism and we can identify the two groups $\bar{\mathcal{U}}$ and $\mathcal{A}(\mathbb{R}^n)$. Consider now a partition of the configuration space $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p}$ and the diagram below:

$$\begin{array}{ccc} \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p} & \stackrel{\Sigma_u}{\longrightarrow} & \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p} \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_p} & \stackrel{\Sigma_u^2}{\longrightarrow} & \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_p} \\ \downarrow \pi_2 & & \downarrow \pi_2 \\ \vdots & & \vdots \\ \downarrow \pi_{p-1} & & \downarrow \pi_{p-1} \\ \mathbb{R}^{n_p} & \stackrel{\Sigma_u^p}{\longrightarrow} & \mathbb{R}^{n_p} \end{array}$$

With this notation, we define a chain of subgroups as

Definition 2 *Let $\bar{\mathcal{U}}_p$ be the subset $\bar{\mathcal{U}}$ of elements $\mathbf{u}$ such that $\Sigma_u^p = (\pi_1 \circ \cdots \circ \pi_{p-1}) \circ \Sigma_u \circ (\pi_1 \circ \cdots \circ \pi_{p-1})^{-1} = Id(\mathbb{R}^{n_p})$. Analogously, define the subset $\bar{\mathcal{U}}_{i-1} \subset \bar{\mathcal{U}}_i$, for $i = p, \ldots, 3$, as the set of elements Σ_u such that $\Sigma_u^{i-1} = (\pi_1 \circ \cdots \circ \pi_{i-2}) \circ \Sigma_u \circ (\pi_1 \circ \cdots \circ \pi_{i-2})^{-1} = Id(\mathbb{R}^{n_{i-1}} \times \cdots \times \mathbb{R}^{n_p})$.*

Observe that for each $i = 2, \ldots, p-1$, $\bar{\mathcal{U}}_i$ is a subgroup of $\bar{\mathcal{U}}_{i+1}$ and $\bar{\mathcal{U}}_p$ is a subgroup of $\bar{\mathcal{U}}$. In the geometric terms of definition 1, at each diagram step there correspond a base space $\mathbb{R}^{n_q} \times \cdots \times \mathbb{R}^{n_p}$, a fiber $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_{q-1}}$, and a set of (independent) closed loop paths on the base $\bar{\mathcal{U}}_q$.

In order to exploit such nested subgroup structure for steering the system (2) in a way similar to what was done for the rolling polyhedra example, it is necessary that the action of $\bar{\mathcal{U}}_q$ is abelian on the sub–fiber $\mathbb{R}^{n_{q-1}}$. This requirement is embodied by the following definition:

Definition 3 *A driftless nonholonomic system (2) is said to be decomposable in symmetric variables if a partition $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p} = \mathbb{R}^n$ and a corresponding chain of control subgroups $\bar{\mathcal{U}}_p \subset \cdots \subset \bar{\mathcal{U}}_2$ as in def. 2, such that $\bar{\mathcal{U}}_q$ is a group with abelian action on $\mathbb{R}^{n_{q-1}}$, $q = 2, \ldots, p$.*

Given a general system such as (2), it might be difficult to decide whether or not it is decomposable in symmetric variables, and to find such decomposition. However, a sufficient condition holds for an important class of systems:

Theorem 1 *If Σ is a control system in strictly triangular form (1), it is decomposable in symmetric variables, the decomposition being exactly that in (1).*

Remark. Clearly, systems that are feedback equivalent to ST form can also be decomposed in symmetric variables, and constitute a very large and important class of nonholonomic systems. Chained form systems are obviously in ST form; nilpotent systems can also be shown to be feedback equivalent to ST form. However, the class of ST system is larger than both chained and nilpotent systems: as an example, the plate–ball system is not chained nor nilpotent, but can be put in ST form by feedback and change of coordinates. A characterization of systems that can be put in ST form by coordinate changes is given in [9]; the study of necessary and sufficient conditions for feedback equivalence to ST form is under way at present.

Proof. We need to show that any control $\mathbf{u} \in \bar{\mathcal{U}}_q$ has abelian action on the variables $\mathbf{x}_{q-1}$. This is proved by showing that for a pair of controls $\mathbf{u}_1, \mathbf{u}_2 \in \bar{\mathcal{U}}_q$, $\Sigma_{\mathbf{u}_1 \circ \mathbf{u}_2}^{q-1} = \Sigma_{\mathbf{u}_1}^{q-1} + \Sigma_{\mathbf{u}_2}^{q-1}$. Consider the last $n_{q-1} + n_q + \cdots + n_p$ variables of the control system (1), and recall that by definition of $\bar{\mathcal{U}}_q$, $\Sigma_{\mathbf{u}_1 \circ \mathbf{u}_2}^{q} = Id(\mathbb{R}^{n_q})$. Due to the ST form, the flow of $\mathbf{x}_{q-1}$ is given by

$$\boldsymbol{\delta}_{q-1}(\mathbf{u}_1 \circ \mathbf{u}_2) = \int_0^{T_1+T_2} \mathbf{g}_{q-1}(\mathbf{x}_q, \ldots, \mathbf{x}_p)\,(\mathbf{u}_1 \circ \mathbf{u}_2)\; dt$$

where $[0, T_1]$ and $[0, T_2]$ are the support sets of $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively. Applying the definition of concatenation of controls, the integral can be split as

$$\boldsymbol{\delta}_{q-1}(\mathbf{u}_1 \circ \mathbf{u}_2) = \int_0^{T_1} \mathbf{g}_{q-1}\mathbf{u}_1(t)dt + \int_{T_1}^{T_1+T_2} \mathbf{g}_{q-1}\mathbf{u}_2(t-T_1)dt$$

and, by changing variables ($s = t - T_1$) in the second term on the right–hand and recalling that $(\mathbf{x}_q, \ldots, \mathbf{x}_p)(T_1) = (\mathbf{x}_q, \ldots, \mathbf{x}_p)(0)$, we get

$$\boldsymbol{\delta}_{q-1}(\mathbf{u}_1 \circ \mathbf{u}_2) = \boldsymbol{\delta}_{q-1}(\mathbf{u}_1) + \boldsymbol{\delta}_{q-1}(\mathbf{u}_2).\text{q.e.d.}$$

The problem of constructing a steering control will be described in the following in two steps. We first give a realization of controls with the property of belonging to the chain of subgroups acting on decreasing subfibers as described in def.3. Secondly, based on a choice of a finite number of such controls, an algorithm to achieve steering is described.

The first problem is solved using the concept of derived flag of a group which we recall in the following definition:

Definition 4 *Let $\mathcal{H}$ be a group then $\mathcal{H}_1 = [\mathcal{H}, \mathcal{H}] = \{h_1 h_2 h_1^{-1} h_2^{-1},\ h_1, h_2 \in \mathcal{H}\}$ is the* derived subgroup *of $\mathcal{H}$. By induction we define $\mathcal{H}_i = [\mathcal{H}_{i-1}, \mathcal{H}_{i-1}]$, $\forall i = 2, \ldots$.*

A subchain of control sets $\mathcal{V}_q$, $q = 2, \ldots p$ such that

$$\begin{array}{ccccccc} \bar{\mathcal{U}} & \supset & \bar{\mathcal{U}}_p & \supset & \cdots & \supset & \bar{\mathcal{U}}_2 \\ & & \cup & & & & \cup \\ & & \mathcal{V}_p & \supset & \cdots & \supset & \mathcal{V}_2 \end{array} \qquad (3)$$

is obtained as follows. Let $\mathcal{V}_p = \bar{\mathcal{U}}_p$ be the set of functions $\mathbf{u} : [0, T] \mapsto \mathbb{R}^{n_p}$ such that $\int_0^T \mathbf{u}(t)dt = 0$. It can be verified that the chain of subgroups $\mathcal{V}_q$ with

$$\mathcal{V}_q = [\mathcal{V}_{q+1}, \mathcal{V}_{q+1}] = \{\mathbf{u} \circ \mathbf{v} \circ \mathbf{u}^{-1} \circ \mathbf{v}^{-1},\ \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}_{q+1}\}$$

for all $q = 2, ..., p-1$ satisfies (3).

In general, the subgroups $\bar{\mathcal{U}}_{q+1}$ of controls that steer variables in $\mathbb{R}^{n_p} \times \cdots \times \mathbb{R}^{n_{q+1}}$ along closed loops, may not possess a finite number of generators. A control in $\bar{\mathcal{U}}_{q+1}$ that obtains a desired change in the symmetric variables $\mathbf{x}_q$ might therefore be searched iteratively, by deforming an initial control loop continuously (this would be similar to the procedure proposed in [16]). Differently, the technique we propose consists in considering only a finite number of (suitable) elements in $\bar{\mathcal{U}}_{q+1}$, and in combining them. This is possible because the action of $\bar{\mathcal{U}}_{q+1}$ on $\mathbb{R}^{n_q}$ is abelian, hence the net motion produced on the symmetric variables can be described by a vector of translation in $\mathbb{R}^{n_q}$.

Let $\boldsymbol{\delta}_q^k, k = 1, \ldots, N_q$ denote the quantum translation of variables $\mathbf{x}_q$ corresponding to some set of N_q controls $\mathbf{u}^k \in \bar{\mathcal{U}}_{q+1}$. Consecutive applications of different quanta will displace the state variables $\mathbf{x}_q$ by

$$\boldsymbol{\delta}_q = \sum_{k=1}^{N_q} a_k \boldsymbol{\delta}_q^k, \quad a_k \in Z \tag{4}$$

Planning for the variables $\mathbf{x}_q$ consists therefore in searching for a solution of the Diophantine equation (4) for a given desired $\boldsymbol{\delta}_q^{des}$.

The set of possible displacements $\boldsymbol{\delta}_q$ for a finite number N_q of control quanta, applied each a finite number of times a_k, is obviously finite. A practical solution has therefore to be considered in an approximate sense:

Definition 5 *Given a tolerance $\epsilon_q > 0$, a set $\{a_1, \ldots, a_{N_q}\} \in Z^{N_q}$ is said to be an ϵ_q–approximated solution to (4) if $\|\boldsymbol{\delta}_q^{des} - \sum_{k=1}^{N_q} a_k \boldsymbol{\delta}_q^k\| < \epsilon_q$. A set of control quanta $\boldsymbol{\Delta}_q = \{\boldsymbol{\delta}_q^k, k = 1, \ldots, N_q\}$ is said to be* exhaustive *in $\mathbb{R}^{n_q}$ if, for any $\epsilon_q > 0$ and any $\boldsymbol{\delta}_q^{des}$, an ϵ_q–approximated solution exists.*

A control quanta set $\boldsymbol{\Delta}_q$ is exhaustive in $\mathbb{R}^{n_q}$ if and only if $N_q \geq n_q + 1$ and there exist $n_q + 1$ elements $\tilde{\boldsymbol{\delta}}_q^j \in \boldsymbol{\Delta}_q$, $j = 1, \ldots, n_q + 1$ such that $\tilde{\boldsymbol{\delta}}_q^j$, $j = 1, \ldots, n_q$ are linearly independent, and

$$\frac{(\tilde{\boldsymbol{\delta}}_q^{n_q+1})^T \tilde{\boldsymbol{\delta}}_q^i}{\|\tilde{\boldsymbol{\delta}}_q^i\|} \in \mathbb{R} \setminus Q, \quad i = 1, \ldots, n_q$$

where Q denotes the set of rational numbers.

If not exhaustive, a set of control quanta will drive the associated symmetric variables to a set of reachable configurations which is a closed proper subset of $\mathbb{R}^{n_q}$. In particular, if $\frac{(\boldsymbol{\delta}_q^i)^T \boldsymbol{\delta}_q^j}{\|\boldsymbol{\delta}_q^i\|} \in Q$, $\forall i, j \in \{1, \ldots, N_q\}$, the reachable set is comprised of discrete points distributed at the vertices of the lattice generated by $\boldsymbol{\Delta}_q$.

Because of the assumed controllability of the nonlinear system (1), a subset of $n_q + 1$ controls in $\bar{\mathcal{U}}_{q+1}$ which generates exhaustive quanta is guaranteed to exist (indeed, this case is generic). However, from a practical point of view, there is little point in insisting to verify such property. In fact, computer representations of numbers are invariably rational, such that, to all practical extents, the computable reachable set is a lattice. To a lattice–generating set of quanta, the Hermite Normal Form algorithm can be applied to reduce the lattice and measure its volume ([15]). An n_q–tuple of lattice generators $X_1, \ldots, X_{n_q}$ is obtained as the non–zero columns of matrix

$$[\mathbf{X}_q \; \mathbf{0}] = \begin{bmatrix} X_q^1 & \cdots & X_q^{n_q} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\delta}_q^1 & \boldsymbol{\delta}_q^2 & \ldots & \boldsymbol{\delta}_q^{N_q} \end{bmatrix} \mathbf{C}_q = \boldsymbol{\Delta}_q \mathbf{C}_q$$

where $\mathbf{C}_q$ is a unimodular integer matrix. The determinant of the Hermite matrix $\det(\mathbf{X}_q)$ represents the area of the mesh of the reachable lattice. From Minkowski's convex body theorem, an ϵ_q–neighborhood of an arbitrary point in $\mathbb{R}^{n_q}$ contains a lattice point if $\pi \epsilon_q^2 \geq 4 \det(\mathbf{X}_q)$.

The control quanta algorithm to steer driftless nonholonomic system in the form (1) is as follows:

Algorithm.

1) Build a set U_p of at least $n_{p-1} + 1$ zero–average controls $\mathbf{u}_k$, and set $q = p - 1$;

2) compute the control quanta set $\boldsymbol{\Delta}_q$ by integrating the flows of the component $\mathbf{x}_q$ in (1).

3) if $\dim \boldsymbol{\Delta}_q < n_q$ (or if the set is ill-conditioned), add one control to U_p and go to step 2);

4) compute the Hermite Normal Form of (the rational approximation of) $\boldsymbol{\Delta}_q$. If the mesh is too coarse, add one control to U_p and go to step 2);

5) if q=2, stop; else, decrease q by one, and go to step 3).

Once the algorithm is ended, the closest vertex of the lattice $\mathbf{X}_q \bar{\mathbf{z}}$ to a desired final configuration is quickly computed by truncation of solutions $\mathbf{z} = \mathbf{X}_q^{-1} \boldsymbol{\delta}_q^{des}$, and the actual combinations of control loops that reach that vertex as $\mathbf{C}_q \bar{\mathbf{z}}$. The same is applied to all $q = 2, \ldots, p$.

Remark. A distinct advantage of the control quanta method over iterative schemes is that in the latter the computational cost of every planning problem is constantly high, while in the former the setup cost only is relevant, while all subsequent solutions are almost trivial. Details on the lattice reduction techniques, and on computational complexity of the method, can be found in [10].

4 Example

Consider the plate–ball system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \hline \dot{x}_3 \\ \hline \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{c|c} -\sin(x_3)\cos(x_5) & -\cos(x_3) \\ \cos(x_3)\cos(x_5) & -\sin(x_3) \\ \hline \sin(x_5) & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

and three controls in $\bar{\mathcal{U}}_{45}$, $\mathbf{u}_1 = [\sin(t), \cos(t)]^T$, $\mathbf{u}_2 = [\sin(t), 2\cos(t)]^T$, $\mathbf{u}_3 = [\cos(t), -2\sin(t)]^T$, all defined on $[0, 2\pi]$. Controls in $\bar{\mathcal{U}}_3$ are $\mathbf{u}_{12} = \mathbf{u}_1 \circ \mathbf{u}_2 \circ \mathbf{u}_1^{-1} \circ \mathbf{u}_2^{-1}$, $\mathbf{u}_{13} = \mathbf{u}_1 \circ \mathbf{u}_3 \circ \mathbf{u}_1^{-1} \circ \mathbf{u}_3^{-1}$, and $\mathbf{u}_{32} = \mathbf{u}_3 \circ \mathbf{u}_2 \circ \mathbf{u}_3^{-1} \circ \mathbf{u}_2^{-1}$, defined on $[0, 8\pi]$. The corresponding control quanta are the columns of

$$\boldsymbol{\Delta}_3 = \begin{bmatrix} 2.7649 & 3.6237 & -1.5080 \end{bmatrix}$$

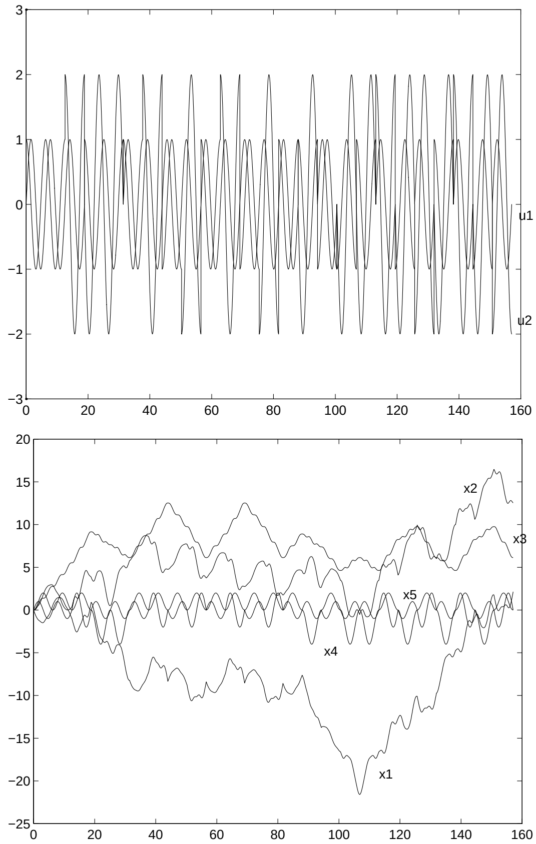

Figure 2: Controls (top) and state evolution (bottom) to steer the plate–ball system in the example

and

$$\mathbf{\Delta}_{12} = \begin{bmatrix} 0.1064 & -12.5385 & 10.7986 \\ -2.0147 & -4.1201 & 8.1733 \end{bmatrix}$$

respectively. A control that brings the system from zero to $[2.0, 12.5, 6.0, 0.0, 0.0]^T$, within tolerance 0.2 is computed as $\mathbf{u}_1 \circ \mathbf{u}_1 \circ \mathbf{u}_2 \circ \mathbf{u}_3 \circ \mathbf{u}_3 \circ \mathbf{u}_{12} \circ \mathbf{u}_{12} \circ \mathbf{u}_{13} \circ \mathbf{u}_{32} \circ \mathbf{u}_{32}$, and is reported in fig. 2. The configuration actually reached by the system is $[2.1353, 12.5428, 6.1379, 0.0001, 0.0000]^T$.

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