Stability and Robustness of Optimal Synthesis for Route Tracking by Dubins' Vehicles

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Abstract

In this paper we consider the properties of stability and robustness of an optimal control synthesis obtained for the problem of route tracking by a kinematic vehicle moving forward only with a lower bounded turning radius. This model, sometimes referred to as "Dubins' vehicle", is relevant to the kinematics of road vehicles as well as aircraft cruising at constant altitude, or sea vessels.

1 Introduction

A feedback control allowing to drive a mobile robot, with a constraint on the turning wheels angle, from a generic configuration to connect to a specified route was described by Souéres, Balluchi and Bicchi, in [8]. A feedback law was proposed, such that straight routes can be approached optimally, while the system is asymptotically stabilized. Experimental results were reported there that seemed to support the capability of the obtained synthesis to remain acceptable in spite of the many discrepancies between the theoretical model used for analytic derivations, and the practical, uncertain and noisy plant.

In this paper, we pursue a theoretical analysis of such capability, more precisely of the stability and robustness properties of the synthesis with respect to model uncertainties and measurement noise.

2 The optimal synthesis

We describe in this section the optimal synthesis stated in [8]. This work characterizes the optimal control law allowing to drive a bounded-curvature vehicle towards a straight route. The model we consider ignores the vehicle dynamics. However, it explicitly takes into account inherent limitations of automobiles along highways and aircraft cruising at constant altitude, thus providing a complete model of the kinematics involved, and a reference framework for extending results to more complex models.

Let the configuration of the vehicle be described as $X = (M, \theta) \in \mathbb{R}^2 \times S^1$, where $M = (x, y)$ are the coordinates of the reference point of the vehicle with respect to a reference frame, θ is the angle representing the vehicle's direction with respect to the frame x-axis. The kinematics of the vehicle is described by

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \omega \end{cases} \qquad (1)$$

where v is the forward velocity of the vehicle and ω its angular velocity. Without loss of generality, up to a time–axis rescaling, we assume that $\dot{v}(t) = 0$, $v(t) \equiv V$. The turning radius of the vehicle is lower bounded by a constant value $R > 0$, which results in an upper bound on the vehicle's angular velocity ω as

$$|\omega| < \frac{V}{R}. \qquad (2)$$

For this model we consider the problem of determining a path of minimal length for reaching tangentially a rectilinear route with a specified direction of motion. We denote by $\mathcal{T}$ a target rectilinear path in the plane, with a prescribed direction of motion determined by the angle $\alpha \in [-\pi, \pi]$ with respect to the x-axis (see figure 1). We consider the optimal control problem:

$$\text{Minimize } J = \int_0^T \sqrt{\dot{x}^2 + \dot{y}^2}dt = VT, \qquad (3)$$

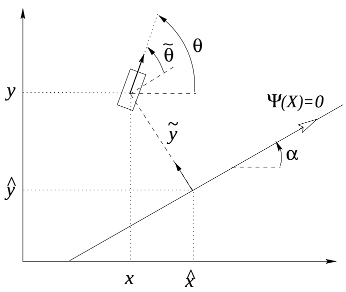

Figure 1: Path–based coordinates along the directed line $\mathcal{T}$.

subject to (1) and (2), with $X(0) = (M_0, \theta_0)$ and such that, at the unspecified terminal time T, $M(T) \in \mathcal{T}$ and $\theta(T) = \alpha$. To describe the optimal controller it is expedient to use path–based coordinates $s, \tilde{y}, \tilde{\theta}$ (see figure 1). s is the curvilinear abscissa measuring the motion of the perpendicular projection of the robot reference point on $\mathcal{T}$. $\tilde{y}$ represents the distance between $\mathcal{T}$ and the robot reference point, divided by R. $\tilde{\theta} = \theta - \alpha$ is the heading angle error. The tracking problem (3) is reformulated in these variables as a minimum–time convergence to the submanifold $\mathcal{S} = \left\{(s, \tilde{y}, \tilde{\theta}) : \tilde{y} = 0, \tilde{\theta} = 0\right\}$. Equivalently, we will refer to the convergence of $(\tilde{y}, \tilde{\theta})$ to the origin $(0,0)$ of the *reduced state space*. $\tilde{y}$ and $\tilde{\theta}$ obey the following equation:

$$\begin{array}{rcl} \dot{\tilde{y}} & = & \sin(\tilde{\theta})\,\frac{V}{R} \\ \dot{\tilde{\theta}} & = & \omega \end{array} \tag{4}$$

In what follows we will only deal with reduced state space coordinates $\tilde{y}, \tilde{\theta}$. The optimal route tracking problem is converted in this case to the problem of stabilizing every point to the origin of the reduced state space in minimum time. The optimal control synthesis is reported in figure 2. Introduce

$$\sigma_N(\tilde{y}, \tilde{\theta}) = \tilde{y} + 1 + \cos(\tilde{\theta}), \tag{5}$$

$$\sigma_P(\tilde{y}, \tilde{\theta}) = \tilde{y} - 1 - \cos(\tilde{\theta}). \tag{6}$$

The optimal feedback control is defined inside the region

$$\mathcal{D}_{(\tilde{y},\tilde{\theta})} = \left\{ \begin{array}{l} \sigma_N(\tilde{y}, \tilde{\theta}) < 0 \ \wedge \ \tilde{\theta} \in [\pi, \frac{3}{2}\pi) \ \vee \\ \sigma_P(\tilde{y}, \tilde{\theta}) \leq 0 \ \wedge \ \tilde{\theta} \in (\frac{\pi}{2}, \pi) \ \vee \\ \tilde{\theta} \in [-\frac{\pi}{2}, \frac{\pi}{2}] \ \vee \\ \sigma_N(\tilde{y}, \tilde{\theta}) \geq 0 \ \wedge \ \tilde{\theta} \in [-\pi, -\frac{\pi}{2}) \ \vee \\ \sigma_P(\tilde{y}, \tilde{\theta}) > 0 \ \wedge \ \tilde{\theta} \in (-\frac{3}{2}\pi, -\pi) \end{array} \right. \tag{7}$$

in the reduced state space $(\tilde{y}, \tilde{\theta})$, which, modulo 2π angles on $\tilde{\theta}$, corresponds to the whole space (see Figure (2)).

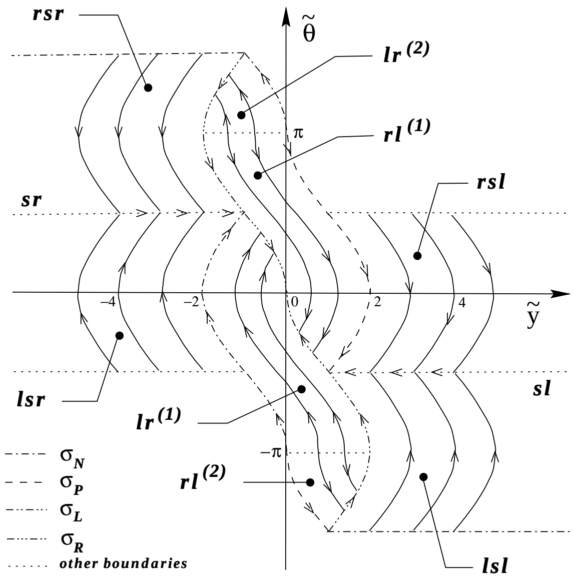

Figure 2: Shortest paths synthesis in the $(\tilde{y}, \tilde{\theta})$ - plane

As the problem we consider is a subproblem of Dubins' problem, extremal arcs are necessarily of two types: arcs of a circle with minimal radius R or straight line segments [4], [9], [5]. Using the hybrid automaton formalism (see [10]), the feedback controller can be described by the three following modes:

$$\begin{array}{lll} \bullet & \textit{go\_straight}, & \text{where } \omega = 0; \\ \bullet & \textit{turn\_right}, & \text{where } \omega = -\dfrac{V}{R}; \\ \bullet & \textit{turn\_left}, & \text{where } \omega = +\dfrac{V}{R}. \end{array} \tag{8}$$

The control mode is selected according to the partition of the domain $\mathcal{D}_{(\tilde{y},\tilde{\theta})}$ in (7) defined as follows:

$$\begin{array}{rcl} \Omega^0 & = & \mathbf{sr} \cup \mathbf{sl} \cup \{(0,0)\}, \\ \Omega^- & = & \mathbf{r} \cup \mathbf{rsr} \cup \mathbf{rsl} \cup \mathbf{rl}^{(1)} \cup \mathbf{rl}^{(2)}, \\ \Omega^+ & = & \mathbf{l} \cup \mathbf{lsr} \cup \mathbf{lsl} \cup \mathbf{lr}^{(1)} \cup \mathbf{lr}^{(2)}, \end{array} \tag{9}$$

where the twelve subsets on the right–hand side of (9) are defined in Table 1 and $\sigma_R(\tilde{y}, \tilde{\theta}) = \tilde{y} + 1 - \cos(\tilde{\theta})$, $\sigma_L(\tilde{y}, \tilde{\theta}) = \tilde{y} - 1 + \cos(\tilde{\theta})$.

The hybrid control is defined by the following rules

$$\begin{array}{lll} \bullet & \textit{go\_straight}, & \text{if } (\tilde{y}, \tilde{\theta}) \in \Omega^0 \\ \bullet & \textit{turn\_right}, & \text{if } (\tilde{y}, \tilde{\theta}) \in \Omega^- \\ \bullet & \textit{turn\_left}, & \text{if } (\tilde{y}, \tilde{\theta}) \in \Omega^+ \end{array} \tag{10}$$

$\mathbf{r} = \mathbf{r}^{(1)} \cup \mathbf{r}^{(1)} \cup \mathbf{r}^{(3)}$ $\mathbf{lr}^{(1)} = \mathbf{lr}^{(1.1)} \cup \mathbf{lr}^{(1.2)} \cup \mathbf{lr}^{(1.3)}$	$\mathbf{lsr} = \mathbf{lsr}^{(1)} \cup \mathbf{lsr}^{(2)}$ $\mathbf{rsr} = \mathbf{rsr}^{(1)} \cup \mathbf{rsr}^{(2)}$
$\mathbf{l} = \mathbf{l}^{(1)} \cup \mathbf{l}^{(1)} \cup \mathbf{l}^{(3)}$ $\mathbf{rl}^{(1)} = \mathbf{rl}^{(1.1)} \cup \mathbf{rl}^{(1.2)} \cup \mathbf{rl}^{(1.3)}$	$\mathbf{rsl} = \mathbf{rsl}^{(1)} \cup \mathbf{rsl}^{(2)}$ $\mathbf{lsl} = \mathbf{lsl}^{(1)} \cup \mathbf{lsl}^{(2)}$
$\mathbf{r}^{(1)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (0,\frac{\pi}{2}), \sigma_R(\tilde{y},\tilde{\theta}) = 0\},$	$\mathbf{l}^{(1)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (-\frac{\pi}{2},0), \sigma_L(\tilde{y},\tilde{\theta}) = 0\},$
$\mathbf{r}^{(2)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in [\frac{\pi}{2},\pi), \sigma_R(\tilde{y},\tilde{\theta}) = 0\},$	$\mathbf{l}^{(2)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (-\pi,-\frac{\pi}{2}], \sigma_L(\tilde{y},\tilde{\theta}) = 0\},$
$\mathbf{r}^{(3)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in [\pi,\frac{3}{2}\pi), \sigma_R(\tilde{y},\tilde{\theta}) = 0\} \cup \{(0,\pi)\}$	$\mathbf{l}^{(3)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (-\frac{3}{2}\pi,-\pi], \sigma_L(\tilde{y},\tilde{\theta}) = 0\},$
$\mathbf{lr}^{(1.1)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (0,\frac{\pi}{2}), \sigma_N(\tilde{y},\tilde{\theta}) \geq 0, \sigma_L(\tilde{y},\tilde{\theta}) < 0\}$	$\mathbf{rl}^{(1.1)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (-\frac{\pi}{2},0), \sigma_L(\tilde{y},\tilde{\theta}) > 0, \sigma_P(\tilde{y},\tilde{\theta}) \leq 0\}$
$\mathbf{lr}^{(1.2)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (-\frac{\pi}{2},0], \sigma_N(\tilde{y},\tilde{\theta}) \geq 0, \sigma_R(\tilde{y},\tilde{\theta}) < 0\}$	$\mathbf{rl}^{(1.2)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in [0,\frac{\pi}{2}), \sigma_R(\tilde{y},\tilde{\theta}) > 0, \sigma_P(\tilde{y},\tilde{\theta}) \leq 0\}$
$\mathbf{lr}^{(1.3)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (-\pi,-\frac{\pi}{2}], \sigma_N(\tilde{y},\tilde{\theta}) \geq 0, \sigma_R(\tilde{y},\tilde{\theta}) < 0\}$	$\mathbf{rl}^{(1.3)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in [\frac{\pi}{2},\pi), \sigma_R(\tilde{y},\tilde{\theta}) > 0, \sigma_P(\tilde{y},\tilde{\theta}) \leq 0\}$
$\mathbf{lr}^{(2)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in [\pi,\frac{3}{2}\pi), \sigma_R(\tilde{y},\tilde{\theta}) > 0, \sigma_N(\tilde{y},\tilde{\theta}) < 0\}$	$\mathbf{rl}^{(2)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (-\frac{3}{2}\pi,-\pi], \sigma_P(\tilde{y},\tilde{\theta}) > 0, \sigma_L(\tilde{y},\tilde{\theta}) < 0\}$
$\mathbf{sr} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{y} < -1, \tilde{\theta} = \frac{\pi}{2}\}$	$\mathbf{sl} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{y} > +1, \tilde{\theta} = -\frac{\pi}{2}\}$
$\mathbf{lsr}^{(1)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in [0,\frac{\pi}{2}), \sigma_N(\tilde{y},\tilde{\theta}) < 0\}$	$\mathbf{rsl}^{(1)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (-\frac{\pi}{2},0], \sigma_P(\tilde{y},\tilde{\theta}) > 0\}$
$\mathbf{lsr}^{(2)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in [-\frac{\pi}{2},0), \sigma_N(\tilde{y},\tilde{\theta}) < 0\}$	$\mathbf{rsl}^{(2)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (0,\frac{\pi}{2}], \sigma_P(\tilde{y},\tilde{\theta}) > 0\}$
$\mathbf{rsr}^{(1)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (\frac{\pi}{2},\pi), \sigma_R(\tilde{y},\tilde{\theta}) < 0\}$	$\mathbf{lsl}^{(1)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (-\pi,-\frac{\pi}{2}), \sigma_L(\tilde{y},\tilde{\theta}) > 0\}$
$\mathbf{rsr}^{(2)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in [\pi,\frac{3}{2}\pi), \sigma_R(\tilde{y},\tilde{\theta}) < 0\}$	$\mathbf{lsl}^{(2)} = \{(\tilde{y},\tilde{\theta}) \mid \tilde{\theta} \in (-\frac{3}{2}\pi,-\pi], \sigma_L(\tilde{y},\tilde{\theta}) > 0\}$

Table 1: Partition of domain $\mathcal{D}_{(\tilde{y},\tilde{\theta})}$.

In Figure 2 the boundaries between subsets of the partition (9) are represented by dotted lines, and the direction of motion is represented by directed curves.

3 Structural stability

In [3], the question has been addressed considering the equivalent problem of reaching all points of the plane from the origin in minimum time (it is enough to take the opposite dynamics and reverse the time on the system). In particular, it was shown that generically the optimal synthesis exists and correponds to a Withney stratification of the plane. The structural stability of optimal synthesis under generic assumptions was also proved. Moreover, all singularities of the synthesis were classified in [6]. Thus we consider the associated control system:

$$\left\{ \begin{array}{rcl} \dot{y} & = & -\frac{V}{R}\sin(\theta) \\ \dot{\theta} & = & \frac{V}{R}u \end{array} \right. , \tag{11}$$

where $|u| \leq 1$ (we drop $\tilde{}$ on y, θ for brevity and replace ω with u). Clearly there is no need to change sign in the right–hand side of the second equation. This system can be also written in the form

$$\dot{x} = F(x) + uG(x), \qquad x \in R^2, |u| \leq 1, \tag{12}$$

with $x = (y,\theta)$, $F(x) = (-\frac{V}{R}\sin\theta, 0)$, $G(x) = (0, \frac{V}{R})$.

Our aim is to prove that the optimal synthesis of figure 2 is stable in the following sense. Every control system of the type (12), corresponding to a couple of vector fields (F', G') that are a small perturbation in the $\mathcal{C}^3$ norm of (F, G), presents the same type of time optimal synthesis, that is the topological properties do not change. In [3] a precise definition of stability is given that concerns the existence of an homeomorphism that sends arcs of one synthesis onto arcs of the other.

Our synthesis is not stable exactly in the same sense because of the following. The point intersection of the regions $\mathbf{rl(1)}, \mathbf{rsl}_{\frac{\pi}{2}}, \mathbf{lsl}_{\frac{\pi}{2}}$, is connected with an optimal arc to the point intersection of the regions $\mathbf{lr(1)}, \mathbf{rsl}_{\frac{\pi}{2}}, \mathbf{lsl}_{\frac{\pi}{2}}$. This feature is not stable. However, this is the only topological feature that can be destroyed by a generic perturbation in $\mathcal{C}^3$. Hence, with the above specification, we will check the stability of our optimal synthesis checking the stability conditions given in [3] and discussing the cases where they are not verified. Therefore we ensure the stability with respect to model uncertainties.

A key role in the analysis of these systems is played by the functions

$$\Delta_A = F(x) \wedge G(x) = -\frac{V^2}{R^2}\sin\theta,$$

$$\Delta_B = G(x) \wedge [F,G](x) = -\frac{V^3}{R^3}\cos\theta,$$

where $[F,G]$ indicates the Lie bracket of the vector fields F and G. In particular, see [3], all optimal trajectories are bang-bang except possibly those that run on the set where Δ_B vanishes. These trajectories are called turnpikes. In the optimal synthesis described above they correspond to the curves **sl** and **sr**. Other special curves that show up are the overlap curves, formed by points reached optimally by two trajectories coming from different direction. In our case these correspond to the intersection of the boundaries of regions: $\mathbf{rl}(\mathbf{1})$ and $\mathbf{lr}(\mathbf{2})$, $\mathbf{lr}(\mathbf{1})$ and $\mathbf{rl}(\mathbf{2})$, $\mathbf{rsl}_{\frac{\pi}{2}}$ and $\mathbf{lsl}_{\frac{\pi}{2}}$ (that is not **sl**), $\mathbf{rsr}_{\frac{\pi}{2}}$ and $\mathbf{lsr}_{\frac{\pi}{2}}$ (that is not **sr**). Other singularities may show up generically, see [7], but they are not present in the above synthesis.

To construct a synthesis one considers first the trajectories from the origin with constant control ± 1, that correspond to the curves **r** and **l**. Then the local synthesis around these trajectories $\gamma^{\pm}$ is understood studying the following functions. The variational equation along $\gamma^{\pm}$ is given by the system

$$\left\{ \begin{array}{c} \dot{v}^{\pm} = \nabla \left(F \pm G\right) \left(\gamma^{\pm}\left(t\right)\right) \cdot v^{\pm}\left(t\right) \\ v\left(t_0\right) = v_0 \end{array} \right. . \qquad (13)$$

In [3], it was introduced the following function: $\vartheta^{\pm}(t) = \arg\left(G\left(0\right), v^{\pm}\left(G\left(\gamma^{\pm}\left(t\right)\right), t; 0\right)\right)$, where $v^{\pm}\left(G\left(\gamma^{\pm}\left(t\right)\right), t; 0\right)$ indicates the solution of (13) with $t_0 = t$ and $v_0 = G\left(\gamma^{\pm}\left(t\right)\right)$. Let us now focus our attention on γ^{+} being the analysis along γ^{-} similar. We have

$$\gamma^{+}(t) = \left(-1 + \cos\left(\frac{V}{R}t\right), \frac{V}{R}t\right). \qquad (14)$$

The function ϑ^{+} is a Morse function (with the restriction $\theta \in [-\pi, \pi]$).

At the conjugate point to the origin, that is when $t = \pi$, with the notation of [3], we have $\delta \neq 0$, that is the stability condition $(SA8)$ is violated. This is due to the fact that the second components of the vector fields F and G do not depend on θ. However, since at that point the vector fields $F+G$ and $F-G$ are parallel but with opposite direction, the local synthesis is stable, being non–generic only the fact that the overlap curve (that in the above synthesis was the intersection of the regions $\mathbf{rl}(\mathbf{1})$ and $\mathbf{lr}(\mathbf{2})$) is parallel to the y axis. Indeed this curve is generically tangent to the trajectory γ^{+} (see [3] for details).

Hence the synthesis near the curves $\gamma^{\pm}$ is stable. The stability of the whole synthesis is obtained checking few other conditions. For example, the control along the singular trajectories is $u = 0$, so a small perturbation of the system does not affect the presence of singular trajectories. Moreover, $\nabla\Delta_B \cdot (F \pm G) \neq 0$ hence the two fields $(F \pm G)$ always point to the opposite side of the set where Δ_B vanishes. There are stability conditions also on the points that are intersetion of special curves, see [3], that can be checked easily.

This concludes the discussion of the structural stability of the optimal synthesis.

4 Robustness

We notice that the optimal synthesis is indeed a Boltianskii-Brunovsky type synthesis on a stratification of the plane, see [7]. We thus have forward uniqueness of Caratheodory solutions to the discontinuous feedback.

Another kind of stability we want to investigate is the one with respect to perturbation of the measured data and the feedback control. In this sense we do not expect all Boltianskii-Brunovsky type synthesis to be stable. However, following the construction of patchy feedbacks recently introduced by Ancona and Bressan [1], it seems natural to assign an order to the cells of the stratified feedback in order to obtain the desired stability. This order is precisely the order in which a stabilizing trajectory should pass through cells and in many cases this is not an exact order.

More precisely, we want to introduce a polygonal patchy feedback arbitrarily "close" to the optimal feedback that is robust for internal measurement error as well as external disturbances, see [2]. Let us introduce precise definitions and state our result.

Definition 1 *A polygonal patch is a pair (Ω, g) where Ω is an open domain with polygonal boundary and g is a smooth vector field, defined in a neighborhood of the closure of Ω, which points inside Ω.*

We say that $g : \Omega \to R^2$ is a polygonal patchy vector fields on Ω if there exists a family of polygonal patches $\{(\Omega_\alpha, g_\alpha), \alpha \in A\}$ such that A is an ordered set, $\{\Omega_\alpha\}_{\alpha\in A}$ is locally finite and $g(x) = g_\alpha(x)$ if $x \in \Omega_\alpha \setminus \cup_{\beta>\alpha}\Omega_\beta$.

The Cauchy problems corresponding to polygonal patchy vector fields admit forward Caratheodory solutions and a unique backward Caratheodory solution (this is exactly the opposite case as for optimal feedback). Given a bounded variation function w, by a solution to

$$\dot{x} = g(x) + \dot{w}, \qquad x(0) = x_0 \tag{15}$$

we mean a measurable function such that

$$x(t) = x_0 + \int_0^t g(x(s))\, ds + (w(t) - w(0)).$$

This kind of perturbations w includes both internal measurement errors and external disturbances, indeed if the perturbed equation is given by

$$\dot{x}(t) = g(x(t) + e_1(t)) + e_2(t)$$

then the function $z(t) = x(t) + e_1(t)$ satisfies the impulsive equation

$$\dot{z}(t) = g(z) + e_2(t) + \dot{e}_1(t).$$

In [2] the following robustness result was proven:

Proposition 1 *Under generic assumptions a bounded polygonal patchy vector fields on R^2, that is uniformly nonzero, satisfies the following. Given $T > 0$ and a compact set K, there exist positive constants C and δ (depending on T and K) such that for every bounded variation function w defined on $[0, T]$ and every solution x to (15) with $x(0) \in K$ there exists a solution y of $\dot{y} = g(y)$ such that*

$$\|x - y\|_{L^\infty} \leq C\ TotVar(w)$$

where $\|\cdot\|_{L^\infty}$ indicates the L^∞ norm and $TotVar$ the total variation.

This results guarantees the robustness with a precise estimate on the error in the uniform norm with respect to the total variation of the disturbance. The optimal synthesis cannot satisfy precisely the same estimate because of the following. The target is a point and the system is controllable at the origin using the first order Lie brackets of the vector fields F and G, not directly with the vector fields $F + uG$, $|u| \leq 1$. This is the reason why the minimum time function is only Hölder continuous of exponent $1/2$ and not Lipschitz. Hence the error in the uniform norm can be estimated only by the square root of the total variation of the disturbance. However, we can associate a polygonal patchy feedback that is arbitrarily "close" to the optimal feedback and satisfy the robustness property of the above proposition.

Definition 2 *Given $\varepsilon > 0$ and a compact set $K \subset R \times S^1$, we say that a polygonal patchy feedback g is ε near (on K) to the optimal feedback $u(x)$ if the following holds. For every Caratheodory solution x to $\dot{x} = g(x)$, with $x(0) \in K$, there exists a time optimal trajectory γ such that*

$$\|x - \gamma\|_{L^\infty} \leq \varepsilon.$$

Our aim is thus to construct a polygonal patchy vector field ε near to the optimal feedback. The idea is the following. Let us use for the optimal feedback the same terminology used above for the synthesis with reversed time. We define some polygonal patchy vector field with patches that correspond to the various strata where the feedback u is constant, namely the one dimensional manifolds corresponding to $\gamma^\pm$ and turnpikes and the two dimensional regions on which $u(x) = \pm 1$. Then we set an order that is compatible with the order in which an optimal trajectory passes trough the strata of the optimal feedback. At the end we obtain the following.

Proposition 2 *For every $\varepsilon > 0$ and K compact there exists a polygonal patchy vector field g that is ε near (on K) to the optimal feedback and satisfies the assumptions of Proposition 1.*

Proof: We start defining the various patches to construct g and then giving the required order.

First we can pick a polygon P contained in a small ball of radius $\delta > 0$ that is neighborhood of the origin. We define a collection of patches that covers the region $K \setminus B$ and this will suffice to reach the conclusion for δ sufficiently small.

We pick a polygonal neighborhood P^+ of the trajectory γ^+ (up to time $3\pi/2$ when γ^+ stops to be optimal). We define the patchy $(\Omega_{\alpha_1}, g_{\alpha_1})$ by setting $\Omega_{\alpha_1} = P^+$ and $g_{\alpha_1} = F + G$. (It is clear that we can choose P^+ so that $F + G$ points always inside on $P^+ \setminus B$ and define the vector field g_{α_1} in a suitable way on B.) In an entirely similar way we define the patchy $(\Omega_{\alpha_2}, g_{\alpha_2})$ corresponding to γ^- and $F - G$.

We then define two patches corresponding to the turnpikes S_1 and $= S_2$, that is the curves **sr** and **sl** respectively. Fix S_1 and let P be a polygonal neighborhood of $S_1 \cap K$. We define $(\Omega_{\alpha_3}, g_{\alpha_3})$ setting $\Omega_{\alpha_3} = P$ and $g_{\alpha_3} = F$. (It is clear that we can choose P in such a way that F points inside $P \setminus (P^+ \cup P^-)$ and define in a suitable way on $(P^+ \cup P^-)$.) In an entirely similar way we define $(\Omega_{\alpha_4}, g_{\alpha_4})$ using a neighborhood of S_2 and again the vector field F.

Then, we can continue defining patches corresponding to polygonal neighborhoods of the regions $\mathbf{lr(1,2)}, \mathbf{rl(1,2)}, \mathbf{rsl}, \mathbf{lsl}, \mathbf{rsr}, \mathbf{lsr}$. Each time, we choose the vector field $F + G$ if on the corresponding region the optimal feedback control $u(x) = 1$, and choose $F - G$ if $u(x) = -1$. It is easy to check that we can ensure the condition of inward pointing of the vector fields (and the collection of patches is clearly finite). Hence it remains to define the order on the set of patches.

We define an order (not total) following the order in which an optimal trajectory passes through the various regions. In particular, we set $\alpha_3 < \alpha_1$ and $\alpha_4 < \alpha_2$. The only additional choice is the one corresponding to overlap curves. Indeed assume that α_i and α_j are the indexes corresponding to the regions neighbooring an overlap curve, then we can set either $\alpha_i < \alpha_j$ or $\alpha_j < \alpha_i$. However, both choices are good. Notice that the order is not total indeed we can not compare α_1 with α_2 or α_3 with α_4.

Choosing the polygonal regions sufficiently small we can ensure that the polygonal patchy vector field g is ε near to the optimal feedback. Moreover, we can define the polygonal regions in order to satisfy the assuptions of Proposition 1. We thus obtain the desired conclusion.

5 Conclusions

In this paper we have considered the properties of stability and robustness of an optimal control synthesis obtained for the problem of route tracking by a Dubin's car. The synthesis was proven to be structurally stable w.r.t. modeling errors by using general tools for two–dimensional optimal control problems from [3]. Robustness w.r.t. to noise in state measurement has been assessed recurring to a recent theory on patchy feedbacks developed in [1]. Besides the relevance of this results in view of the practical applicability of optimal feedback control to several types of vehicles, this work is probably one of the first detailed applications of the above cited results to a complete optimal synthesis.

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